MATHEMATICAL EXCURSIONS

Side trips along paths not generally traveled in elementary courses in Mathematics

BY

HELEN ABBOT MERRILL

DOVER PUBLICATIONS, INC.

This new Dover edition first published
in 1957, is an unabridged and unaltered
republication of the first edition.

Library of Congress Catalog Card Number 57:-14908
Printed in the United States of America.

CONTENTS

ACKNOWLEDGMENT

The problems in this book have been collected over a long period of years with no thought of making public use of them. It is clear to all who gather such material for private use that an acknowledgment of one's debt in detail is quite impossible. The standard books on mathematical recreations in English, French, German, and Italian have been drawn on, as well as the more elementary magazines.

Those who are trying to help our young folks to see not only that Mathematics is a useful tool and a fine mental discipline, but that hard work may be rare good fun, and that the subject widens out into fields of ever growing wonder and fascination, have generally learned what a large rôle entertaining problems may play.

For much help given to me in my teaching and in the preparation of this book I desire here to express my gratitude to a long line of authors whose names must remain unmentioned.

INTRODUCTION

There is something about a puzzle which appeals to almost everyone, young or old. Perhaps it is the challenge to our thinking powers, the feeling that we must not be conquered by so small a thing. Perhaps our curiosity is aroused to see what mode of attack will succeed, by what clever device a puzzle may be solved, especially if its real nature is skilfully concealed.

Any good text-book in Arithmetic or Algebra or Geometry is sure to contain some stimulating problems, but many more such problems, interesting and amusing, as well as instructive, lie a little off the beaten track.

Our course through the early years of mathematical study is apt to be rather clearly mapped out, resembling the plan of a European tour — so many days for London, so many days for Oxford, etc.; quite the right sights to see, but many tourists come home with no notion of the delightful sights which they have not seen for lack of time or lack of guidance.

This little book is meant to play the part that some of the more detailed or specialized guide books play for the tourist. You have made a trip through Arithmetic, but here are some sights you may have missed, something new about counting or division or decimals. Here are a few side trips, away from the main traveled roads, into some of the fields that border on your path through Algebra and Geometry. They are not the chief sights of such a journey, but they may add some pleasure and profit to your trip.

Next to an abundance of poetry stored in our minds, I believe that there are few things that add more to our ability to divert and enjoy ourselves than a good supply of mathematical puzzles.

Perhaps you have heard that Lewis Carroll, author of "Alice in Wonderland," was a mathematician. He was a poor sleeper, and used to amuse himself when wakeful at night by working out problems in his head. Some of these midnight amusements were published in a little book called "Pillow Problems." We might be made wider awake rather than soothed by thinking out problems in the dark, but it is a fine thing to have some on hand for entertainment while one waits for a train or a dentist's appointment, or is shut in with a cold.

A professor of Zoölogy once told me that, if she were shut up in prison and could have only one book, she thought she would ask for a book of Geometry originals, because they would give her entertainment and keep her brain from getting dull.

Most of the mathematical books which are not text-books are written for rather learned people, but this book is not written for the learned. All the mathematical knowledge that it calls for is some Algebra and Geometry, enough to show you that those subjects can be very entertaining. One pleasant feature of Mathematics is that we do not have to know a great deal about it in order to get amusement from it, and a still more pleasant one is that the farther we go the more we find to surprise and entertain us.

There are many different kinds of problems in this book. Anyone who has an interest in such subjects is sure to find something here to prove diverting, to furnish mental exercise, and to give a little notion of what may be found farther on along the mathematical road.

As I take up the various topics I mean to talk to you very informally, as if you were right here with me, putting questions to you which I hope you will try to answer for yourselves before you go on to see what I say in answer.

Perhaps some of the problems will prove to be posers. An occasional hint has been given, but it seems a pity to give too many and so to spoil your pleasure in wrestling with these puzzles. Answers to many of the problems are given, but on a different page, so that you need not see them unless you wish.

These pages contain only a few samples of what is to be found in the inexhaustible storehouse of Mathematics.

MATHEMATICAL EXCURSIONS

CHAPTER I

ON DIVIDING

It is usual in Mathematics to say that one number is divisible by another when after division there is no remainder. Of course, any number may be divided by any other number, but in most cases the result is either a fraction or a mixed number. We shall use the word *divisible* in its usual sense of *exactly divisible.*

We are going to ask how we can tell without going through with the process of division whether a number is divisible by 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13.

A principle that we need to use is that, if each of two numbers is divisible by a certain number, their sum and their difference are both divisible by that number.

For example, 42 and 56 are both divisible by 7, and so are their sum, 98, and their difference, 14.

Or, in general, if both a and b are divisible by c, so are $a + b$ and $a - b$.

Divisibility by 2. A number is divisible by 2 if its last digit is divisible by 2.

Take, for instance, a general number of three digits, $100\,a + 10\,b + c$. Whatever the values of a and b may be, the first two terms are divisible by 2, and so, if c is divisible by 2, the general number is.

Divisibility by 3. A number is divisible by 3 if the sum of its digits is divisible by 3.

This may be proved algebraically for a number of any size. Take, for example, $1000a + 100b + 10c + d$, and let $a + b + c + d$ be divisible by 3.

$$\begin{array}{rrrrl} 999a + & 99b + & 9c & & \text{is divisible by 3,} \\ a + & b + & c + & d & \text{is divisible by 3, and} \\ \hline 1000a + & 100b + & 10c + & d, & \text{their sum, is therefore divisible by 3.} \end{array}$$

Is 722 divisible by 3? No, since $7 + 2 + 2 = 11$.

Is 258 divisible by 3? Yes, since $2 + 5 + 8 = 15$.

It is also true that, if any number is divided by 3, the remainder is the same as when the sum of the digits is divided by 3. For example, when either 722 or 11 is divided by 3, the remainder is 2. It is interesting to notice that the same conclusion is reached when the sum of the digits of 11 is divided by 3.

For example, what is the remainder when 735,869,542 is divided by 3? The sum of the digits is 49, the sum of the digits of 49 is 13, the sum of the digits of 13 is 4, and 4 when divided by 3 leaves a remainder of 1. Hence the original number when divided by 3 leaves a remainder of 1.

Divisibility by 4, 8, 16, etc. Since 100 is divisible by 4, any multiple of 100 is divisible by 4, so that it is merely a question of the divisibility of the number made up of the last two digits.

For instance, is 57,372 divisible by 4? Yes, since 72 is divisible by 4.

Reasoning in the same way, since 1000 is divisible by 8, any number is divisible by 8 if the number made up of the last three digits is divisible by 8. This principle may be carried as far as we please for powers of 2.

Divisibility by 5 or 10. This rule is so well known that it is unnecessary to state it. But do you notice that it is almost exactly like the rule for 2, and do you see the reason why?

Divisibility by 6 and 12. This rule you can easily find for yourselves by noting the factors of 6 and 12.

Divisibility by 9. The test for 9 is like that for 3. A number is divisible by 9 if the sum of its digits is divisible by 9. Can you prove it?

For example, 8577 is divisible by 9, since $8+5+7+7$ is divisible by 9. Instead of the sum of the digits, 27, the sum of the digits of this last number, 9, may be used.

Also 825 is not divisible by 9, and when it is divided by 9, it gives a remainder of 6, since 15, the sum of its digits, gives a remainder of 6 when divided by 9. Further, 6 is the sum of the digits of 15.

Divisibility by 7, 11, and 13. A rule for 7 is less widely known than these other rules, but there is a good way to test at the same time or 7, 11, and 13, depending upon the fact that the product of these three numbers is 1001. Let us first use the test on a special number, 173,299. Since 1001 is divisible by 7, to use only one of its three factors, 1001 × 173, or 173,173, is divisible by 7. If then the difference between this number and 173,299 is divisible by 7, this last number must also be divisible by 7.

Let us write it out in full.

(1)	173,173	is divisible by 7.
(2)	173,299	is to be investigated.
(3)	126	is divisible by 7.

But since (1) and (3) are each divisible by 7, their sum, (2), must also be divisible by 7.

If now we substitute 11 or 13 throughout this discussion for 7, we find that 126 is not divisible by 11 or 13, and so the sum of (1) and (3) cannot be divisible by either.

You see that in a six-figure number this test amounts to finding the difference between two numbers, one made up of the first three, the other of the last three digits, and seeing whether this difference is divisible by 7 or 11 or 13. If the number has only four or five digits, take the difference between the number made up of the last three digits and the one made up of the other one or two.

For example, is 85,176 divisible by 7 or 11 or 13? $176 - 85 = 91$, which is divisible by 7 and 13, not by 11.

Again, 8151 is divisible by 11 and 13, not by 7, since $151 - 8 = 143$.

Numbers of more than six digits may be tested by blocking them off in threes from the right end, and alternately adding and subtracting these smaller numbers.

For example, to test 78,362,495, we have only to reckon $495 - 362 + 78$, and see whether this is divisible by 7 or 11 or 13.

This last form of the rule can be proved easily, and I shall use a number of nine digits or less. Let the number made up of the last three digits be represented by c, that made up of the next three digits by b, and that made up of the remaining digits by a. Then the number N is equal to $1{,}000{,}000\,a + 1000\,b + c$. This puts a into seventh place, or farther if it has more than one digit in it, the three digits of b will occupy the sixth, fifth, and fourth places, and c will fill the three end places. To illustrate,

$1{,}000{,}000\,a + 1000\,b + c$ is being tested.

Suppose that $a - b + c$ is divisible by 7 or 11 or 13.

Then $999{,}999\,a + 1001\,b$ is divisible by that same number.

For,
$$999{,}999 = 3^3 \times 7 \times 11 \times 13 \times 37,$$
$$1001 = 7 \times 11 \times 13,$$

and so, if $a - b + c$ is divisible by 7 or 11 or 13, the given number must be divisible by it.

This test for 7, 11, and 13 is very easily applied to some numbers. Take the following as examples. You can write others as easily tested.

135,245, 630,645, 708,695, 99,127, 524,538,
361,373, 986,426, 783,761.

You can surprise friends by using this test, and seeing if they can guess how you do it. For example, an automobile with five or six figures on its license plate is standing by the curb. After a casual glance, you remark that the number is divisible by 7 and 11, but not by 13. Try it and see what happens. Your friends will perhaps say, "What of it?" But very likely they will be curious to know how you did it.

Another test for 11. For 11 there is a still simpler test. Combine the digits of the number in order, taking them with signs first + then −. If the result is 0 or a multiple of 11, the number is divisible by 11.

For example, is 7293 divisible by 11? Yes, since $7 - 2 + 9 - 3 = 11$.

You can prove this by taking a general form like $1000\,a + 100\,b + 10\,c + d$, and assuming that $a - b + c - d$ is either 0 or a multiple of 11. Then combine the two so that d disappears. You will find that all the coefficients are divisible by 11.

Lewis Carroll tells in his diary of finding a method by which he could tell by inspection not only the remainder when a number is divided by 9, but also the quotient. The method is not given, but it might be interesting to you to see whether you can find such a method.

Now we shall ask how we can write a general expression for a number which is divisible by 2 or 3 or 4, etc. You can see that, if n stands for any integer, $2\,n$ means a number divisible by 2, that is, an even number, $3\,n$ means a number divisible by 3, etc.

Is there any convenient way to represent a number which is not divisible by 2, that is, is odd? We know that, if 1 is added to or subtracted from an even number, the result is always an odd number, and so $2n + 1$ or $2n - 1$ stands for an odd number. This means that every positive integer belongs to one or the other of the two classes, $2n$ or $2n - 1$ if $n = 1, 2, 3$, etc.

In the same way we can make up a general form for all numbers according as they are or are not divisible by 3, for $3n$ represents all numbers that are divisible by 3, and $3n - 1$ and $3n + 1$ all numbers not divisible by 3. If we wish, we may take the two numbers that follow a multiple of 3 instead of the two that lie on either side of it, that is, instead of $3n - 1$ and $3n + 1$ we may choose $3n + 1$ and $3n + 2$.

So for 4; $4n$ stands for all numbers divisible by 4, while $4n + 1$, $4n + 2$, $4n + 3$ stand for all numbers not divisible by 4. Different selections may be made for the last three forms, such as $4n - 1$, $4n + 1$, and $4n + 2$, or $4n - 2$, $4n - 1$, $4n + 1$.

How many forms are needed to represent all numbers that are not divisible by 5 or by 6? What different selections may be made for these forms? How many forms will represent all numbers not divisible by 9? or, in general, by k, any integer?

In general it is best to choose those forms which use the smallest numbers; for instance, $5n - 2$, $5n - 1$, $5n + 1$, $5n + 2$ are apt to be easier to use than $5n + 1$, $5n + 2$, $5n + 3$, $5n + 4$, but it makes no very great difference.

Prime numbers. A number which has no divisors except itself and 1 is called a prime number. Others are called composite, because they are composed or put together by combining different factors. Thus the factors of 13 are only 1 and 13, but $6=1\times6=2\times3$. How many of the first twenty numbers are prime?

Many mathematicians have tried to work out formulas which will give prime numbers. One of these is $n^2 + n + 11$. If you substitute values for n, letting it successively be equal to 1, 2, 3, 4, etc., you will find that you get quite a list of prime numbers. It can be proved without much difficulty that, no matter how many numbers you may substitute for n, even into the thousands or millions, you will get no result that is divisible by 2 or 3 or 5 or 7, and so, of course, no result that is divisible by anything less than 11. If n is a multiple of 11 ($n = 11\,k$), the result is divisible by 11, and also if n is 10, 21, 32, etc. ($n =$?).

The expression $n^2 + n + 17$ is rather like this one for producing primes when different numbers are substituted for n. Substitute any value you please, you will never find a result that can be divided by any number less than 17.

Some have tried to find prime numbers that differ by 2. The only case of three such numbers in a row is 3, 5, 7 (or, if 1 is included, 1, 3, 5). Other "twin primes," as they are called, are 11 and 13, 17 and 19. You can easily add to this list. The farther out we go, the less often they occur. There are none at all between 700 and 800 and between 900 and 1000.

Sometimes you will see the statement that two numbers are "prime to each other." This means that they have no common factor except 1. For instance, 12 and 35, neither of which is a prime number, are prime to each other, since 12 is not divisible by 5 or 7, nor 35 by 2 or 3.

To reduce a fraction to its lowest terms amounts to changing its form so that numerator and denominator are prime to each other.

Here is a trick that is explained by some of the things in this chapter. A gives three dice to B and says: "Put these in any order, and write down the top numbers, then write after them in the same order the numbers that are on the opposite faces."

Suppose B arranges them thus:

then he writes 254,523. Now A says: "Divide this six-digit number by 37. Divide this quotient by 3, and tell me the result. I will tell you how you arranged them without multiplying at all."

$254{,}523 \div 37 = 6879,$

$6879 \div 3 = 2293,$ the number which B reports to A.

Whatever this four-digit number may be, A subtracts 7 from it, divides the remainder by 9, and gets a three-digit number, which gives in the right order the three numbers first written down by B.

In this case A is given the number 2293. He subtracts 7, getting 2286. Dividing by 9, he has 254, and the numbers originally chosen by B were 2, 5, and 4. This trick is rather easy to do, but not easy to guess.

PROBLEMS

1. Are the following numbers divisible by 3? 731, 825, 1674, 48,572.

2. Are the following numbers divisible by 4? by 8? 7648, 96,464, 733,136, 92,592, 83,754.

3. What numbers less than 7 will divide the following? 913,428, 72,524, 9255.

4. What numbers less than 16 will divide the following? 973,822, 373,835.

5. The following numbers are each divisible by at least two numbers less than 16. Find such divisors. 8365, 543,907.

6. If two numbers have the same digits, but in opposite order, prove that their difference is always divisible by 9. For instance, try 8367 and 7638.

7. If b has an even number of digits, and c has the same digits, but in opposite order, prove that $b + c$ is always divisible by 11.

8. Prove that $n(n^2 + 1)$ is always even when n is any integer.

9. Take any prime number greater than 3 and show that it belongs to one of the two following classes:

a. If 1 is added, the sum is divisible by 6; for example, 11, 29, 83.

b. If 1 is subtracted, the difference is divisible by 6; for example, 13, 43, 61.

10. It is easy to give a proof of the statement in No. 9, which, in other words, states that every prime number has the form $6n + 1$ or $6n - 1$. What five forms represent all numbers not divisible by 6? Which of these can always be factored? The others must be the only possible primes. Of course $6n + 1$ is not always a prime number. If, for instance, $n = 4$, $6n + 1 = 25$. But if a number is prime, it can always be written in one or the other of the two forms, $6n + 1$ or $6n - 1$.

11. Prove that 1, 3, 5 and 3, 5, 7, are the only instances of three consecutive odd numbers that are prime.

12. Show that the sum of two twin primes greater than 3 is always divisible by 12.

13. If a fraction is in lowest terms, its square, cube, fourth power, etc., are also in lowest terms.

14. If an even number when divided by 3 leaves a remainder of 2, its half, when divided by 3, leaves a remainder of 1.

15. If an even number when divided by 3 leaves a remainder of 1, what does its half leave when divided by 3?

16. Show that the product of any three consecutive integers is always divisible by 3, that the product of any four consecutive integers is always divisible by 4, and that, in general, the product of any n consecutive integers is always divisible by n.

17. The number 1234 is not divisible by 11. How many numbers divisible by 11 can be formed by changing the order of the four digits? For instance, 1243 is divisible by 11, and you will find that there are seven others.

18. The number 12,345 is not divisible by 11. How many numbers divisible by 11 can be formed by changing the order of the five digits? Try in the same way the numbers 123,456, 1,234,567, 12,345,678.

19. If a cube is not a multiple of 9, the remainder when it is divided by 9 is either 1 or 8.

20. If n is not divisible by 7, either $n^3 - 1$ or $n^3 + 1$ is divisible by 7. For example, $2^3 = 8$, $8 - 1$ is divisible by 7; $3^3 = 27$, $27 + 1$ is divisible by 7. It is not hard to prove this statement by taking general forms for all numbers not divisible by 7, cubing them and either adding or subtracting 1.

21. The sum of all the factors of 120, including 1 but not 120, is 2×120.

22. The sum of all the factors of 672, including 1 but not 672, is 2×672.

23. The sum of all the factors of 6 is 6,

$$6 = 1 \times 2 \times 3 = 1 + 2 + 3.$$

The mathematicians of early Greece called such numbers perfect — the product of all their factors, including 1, is equal to the sum of the factors. Show that 28 is a perfect number, also 496. The next perfect number is 8128, and the next one contains eight digits. All these perfect numbers end in 6 or 28, but no one knows whether this is true of all such numbers, or how many perfect numbers there are all together.

CHAPTER II

DIFFERENT WAYS OF WRITING NUMBERS

Perhaps you have thought sometimes how lucky it is that you do not have to write numbers as the Roman boys and girls did. Even addition looks hard in such a form as this:

DCXIV
CCCXLI

And one would be quite at a loss in trying to subtract or multiply when numbers are written in this form, while division would be far worse.

As a matter of fact, the Romans did not use these written numbers in actual computation, but had a method of calculating with pebbles, the Latin word for which is *calculi*.

Why is it that our modern method is so much better? Why can we readily use the written numbers in all the usual combinations, addition, subtraction, etc.?

There are two special reasons: one is that we have a zero, which is most useful in many ways; the other is that the value of a number depends on the position of its digits. In the Roman notation V, for instance, always meant 5 wherever it might be placed, as in XV or XVIII. But in 85 and 85,111 the first 5 is really treated as 5, while the second 5 is really 5000, that is, changing the position of a digit changes its value.

Just what does 329 mean? Notice that there is no special reason why it must mean one combination of 3, 2, and 9 rather than another. When the Romans wrote MDC, they understood it to mean M + D + C. When we see *abc* in a problem in algebra, we understand it to mean $a \times b \times c$. Now 329 means neither $3 + 2 + 9$ nor $3 \times 2 \times 9$. It is not so simple. It means $3 \times 10^2 + 2 \times 10 + 9$, that is, it is the sum of the digits, each multiplied by a power of 10 that increases by 1 as we move one space to the left. The final digit is multiplied by 10^0, the next by 10^1, etc. The usefulness of zero is seen in such a number as 32,009, which means $3 \times 10^4 + 2 \times 10^3 + 9$. In a more general form, if a number consists of four digits in the order a, b, c, d, the number is written $1000\,a + 100\,b + 10\,c + d$.

This method of writing numbers, known as the Hindu-Arabic method, was introduced into Italy in the 13th century, but in many parts of Europe the Roman numerals were used until the 16th or 17th century.

Now let us carry a little further the principle on which this method depends. Why should powers of 10 be used in these numbers rather than 8 or 12 or some other number? Why should not 526 mean $5 \times 8^2 + 2 \times 8 + 6$? That is, why is 10 chosen rather than some other number? The answer is that other numbers might perfectly well be used instead of 10; in fact, it is doubtless true that, if men had six fingers on each hand, 526 would mean $5 \times 12^2 + 2 \times 12 + 6$. But if they had four fingers on each hand, 526 would mean $5 \times 8^2 + 2 \times 8 + 6$. As a matter of fact, any positive integer except 1 might be used as a base in writing numbers, and it is interesting to experiment with different bases.

What does 283 written to the usual base 10 become when 5 is taken as the base? You have only to remember what 283 really means to find what must be done. What is the highest power of 5 that is less than 283? Evidently 125. $283 = 2 \times 5^3 + 33$. What is the highest power of 5 that is less than 33? $33 = 5^2 + 8$. Again $8 = 5 + 3$. So that 283 to the base 5 is written $2 \times 5^3 + 5^2 + 5 + 3$. Or, omitting each 5, just as we ordinarily omit 10, the number is 2113. It is easy to check this result.

You will see that, while this way of changing the base is easy to follow, the work can be done quite fast by division, as will now be done.

$$\begin{array}{r|l l r}
5 & 283 & & \\
5 & 56 + 3 & \textit{i.e.}\ 283 = & 56 \times 5 + 3 \\
5 & 11 + 1 & 56 = & 11 \times 5 + 1 \\
 & 2 + 1 & 11 = & 2 \times 5 + 1
\end{array}$$

Hence this number is 2113 to the base 5. You see that this amounts to writing the remainders in order, beginning with the last, after carrying the division as far as possible.

As another illustration, let us find to the base 5 the date of the discovery of America.

$$\begin{array}{r|r r}
5 & 1492 & \\
5 & 298 & 2 \\
5 & 59 & 3 \\
5 & 11 & 4 \\
 & 2 & 1
\end{array}$$

The required date is therefore 21,432.

Proof: $2 \times 5^4 + 5^3 + 4 \times 5^2 + 3 \times 5 + 2 = 1492.$

If men had four fingers on each hand, show that the date of the discovery of America would probably be 2724.

When 5 is taken as the base, the only digits needed are those less than 5, just as, when 10 is the base, no digits beyond 9 are needed. To the base 5 the first twelve numbers are written as follows.

1, 2, 3, 4, 10, 11, 12, 13, 14, 20, 21, 22.

Of course, 10 means the base and no units, 11 means the base and 1 unit, 20 means 2 times the base, etc. At first this seems queer, but if we had always been used to it, it would seem perfectly natural, and one would think it strange to use any other base.

The larger the base, the more digits are needed. For the base 12 two must be added to stand for 10 and 11. As a base 12 has certain advantages over 10, since it has more factors, and there are many who believe that it would be worth while to make the change to the "duodecimal system." But if you think of all the books, instruments, charts, etc., in which numbers are used, you can see that such a change would be very upsetting for many years.

The need of two more digits for the duodecimal system appears if we try to write several numbers to the base 12. For example, $1492 = 10 \times 12^2 + 4 \times 12 + 4.$ Suppose X is chosen as a single symbol to stand for 10. This number then becomes $X\,44$.

Write 1709 to the base 12. The result is $11 \times 12^2 + 10 \times 12 + 5$. If we let E represent 11, this becomes $EX\,5$.

If a number in the decimal system ends in 0, we know it is divisible by 10, and therefore by 2 and 5. If a number in the duodecimal system ends in 0, it is divisible by 12, and therefore by 2, 3, 4, and 6.

In some advanced work it is often helpful to write numbers in what is called the "dyadic system," in which the base is 2. Of course, no digit beyond 1 is needed. The smaller the base, the larger the form of a given number. While 1492 to the base 12 is $X\,44$, it becomes, when 2 is taken as the base, 10,111,010,100. Do you see what the two zeros on the end show about its divisibility?

In the dyadic system the first twelve numbers are

1, 10, 11, 100, 101, 110, 111, 1000,
1001, 1010, 1011, 1100.

It would be very easy to make mistakes in copying numbers that are so much alike, and the size of the numbers makes them awkward to use. But if, instead of 25 cents we should write 11,001 cents, just think how wealthy we should feel with so much money!

I hope you see now that the only reason why it is easy to multiply and divide by 10 is that 10 happens to have been chosen as the base of our number system. If 8 or 12 or some other number had been chosen, it would have had the same pleasant character that we now associate with 10. We should divide or multiply by it merely by shifting a point or by annexing a zero.

This point, by the way, should not in this case be called the decimal point, but perhaps the duodecimal or octonal point.

We know that 9 and 11 have some special properties, as, for example, those proved in Chapter I, but we shall now show that these properties are due to the fact that they differ from the base by unity.

If b, an integer greater than 2, is the base, a number is divisible by $b - 1$ if the sum of its digits is divisible by $b - 1$.

Suppose the number has four digits, p, q, r, s, then it will be written $pb^3 + qb^2 + rb + s$. Subtract from this $p + q + r + s$, which by hypothesis is divisible by $b - 1$.

(1) $pb^3 + qb^2 + rb + s$ is to be investigated,
(2) $p \quad + q \quad + r \quad + s$ is divisible by $b - 1$,
(3) $p(b^3 - 1) + q(b^2 - 1) + r(b - 1)$ is divisible by $b - 1$.

Therefore (1), which is the sum of (2) and (3), is divisible by $b - 1$.

In the same way you can prove that, if a number is written to the base b, and the number obtained by alternately adding and subtracting its digits in order either is zero or is divisible by $b + 1$, the number itself is divisible by $b + 1$.

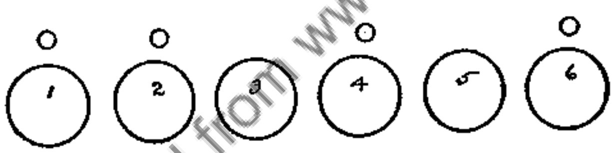
Begin as in the last proof:

$$pb^3 + qb^2 + rb + s \text{ is to be investigated,}$$
$$p \quad - q \quad + r \quad - s \text{ is divisible by } b + 1.$$

Should you add or subtract?

Do you see why in the first of these two proofs we said that the base must be greater than 2? Does it make any difference in the second proof if the base is equal to 2, or is the statement still true when $b = 2$?

There are several tricks that depend on the dyadic system of writing numbers. Here is one of them, for which you will need several small objects — coins or cards or checkers will answer — and some saucers. If the number of objects is not more than 60, you will need only 6 saucers. The trick is to guess the number of coins, say, that some one has chosen, and has arranged according to your instructions while you are out of the room. This person is to take any number of coins he chooses; if the number is odd, he puts 1 just beyond the first saucer, and half the remainder in that



saucer, the rest in a box. Then he takes the coins in the saucer, and, if the number is odd, he puts 1 just beyond the second saucer, half the remainder in that saucer, and half in the box. This will be made clearer by a figure. Suppose he takes 43 coins. Then he puts 1 beyond the first saucer and 21 in it. Since this number is odd, 1 is placed beyond the second saucer and 10 in it, 5 in the third saucer, 1 beyond the fourth saucer and 2 in it, 1 in the fifth saucer, and finally, 1 beyond the sixth saucer. This disposes of all the coins, and the person who is to guess the number is then summoned. Since the coins are laid above the saucers numbered 1, 2, 4, 6, he has only to compute $1 + 2 + 2^3 + 2^5 = 43$, which is the number chosen. Since 39 coins have been put in the box, it would seem to anyone that there is very little to guess from. But you will see that what has been done in arranging the coins is really the same thing that you have been doing in turning an ordinary number into a dyadic number, only it has been greatly camouflaged, so that it is not easily recognized.

PROBLEMS

1. Write to the base 3 the first twenty integers, also 30, 62, 100.

2. In the decimal system $\frac{1}{10}$ is written .1. Suppose that we agree to write $\frac{1}{12}$ as :1. Then $\frac{5}{12}$ will be written :5, etc. With this understanding write as duodecimals $\frac{1}{6}$, $\frac{1}{4}$, $\frac{1}{3}$, $5\frac{2}{3}$.

3. Show that an even number (that is, one divisible by 2) does not in all systems of notation end in an even digit. Try writing 116 to the base 5. Can you find other numbers to illustrate this?

4. Does an odd number necessarily end in an odd digit in all systems of notation?

5. If the base is 10, 234 is divisible by 13. Show that this is not true if the base is 5 or 6 or 7. That is, $2 \times 10^2 + 3 \times 10 + 4$ is divisible by $10 + 3$, but $2 \times 5^2 + 3 \times 5 + 4$ is not divisible by $5 + 3$.

6. If the base is any number greater than 3 (why can it not be less?), show that 231 is divisible by 11 and by 21. This can be proved by taking a as the base, 231 becoming $2a^2 + 3a + 1$.

7. The following statements are true if the base is 10. Are they true to other bases? 156 is divisible by 13; 102 is divisible by 34.

8. Whatever the base may be, 10,101 is divisible by 111.

9. If any base greater than 4 is used, 40,001 is divisible by 221.

10. If any base greater than 2 is used, 121 is a perfect square.

11. If any base greater than 3 is used, 1331 is a perfect cube.

12. If any base greater than 6 is used, 14,641 is a perfect square.

13. If any base greater than 9 is used, 10,009 − 300 is divisible by 133.

NOTE. The next problems, though interesting, seem to have little connection with systems of notation. Their relation to the subject of this chapter will be shown later.

14. A druggist wishes to know what is the smallest number of weights that will enable him to weigh articles of any integral number of ounces from 1 to 40. How many weights does he need, and how much does each one weigh?

15. How does it alter the results in No. 14 if the druggist uses a balance, and can put one or more weights in with the object he is weighing? For instance, if a 3-ounce weight with the package that is being weighed exactly balances a 7-ounce weight, the package must weigh 4 ounces.

16. With five 1-ounce weights, five 10-ounce weights, five 100-ounce weights, etc., show that one can weigh with a balance an object whose weight is any integral number of ounces.

CHAPTER III

MULTIPLYING WITHOUT THE MULTIPLICATION TABLE

Multiplication was carried on for years before any multiplication table was thought of. To multiply was at first only to add a number to itself a certain number of times. To multiply 6 by 3 meant to add three sixes, $6 + 6 + 6 = 3 \times 6$. But after a time it was noticed that it would save a lot of trouble if some of these results of continued addition were kept on hand for use, and so the multiplication table came into existence. Evidently the product of any two integers can be found by addition. But, while it would be quite possible to multiply 465 by 342 by writing 465 down 342 times and adding the column, it would be a very slow process, and one thing which mathematicians particularly like is short cuts. No long pieces of computation for them if they can shorten the work. You will see on the next page how this piece of work can be shortened if we suppose that one knows how to multiply by 10 and 100 only.

This is better than adding a column of 342 numbers, but it probably makes us glad that we know the multiplication table.

$$
\begin{array}{r l}
\left.\begin{array}{r} 46{,}500 \\ 46{,}500 \\ 46{,}500 \end{array}\right\} & = 300 \times 465 \\
\left.\begin{array}{r} 4{,}650 \\ 4{,}650 \\ 4{,}650 \\ 4{,}650 \end{array}\right\} & = 40 \times 465 \\
\left.\begin{array}{r} 465 \\ 465 \end{array}\right\} & = 2 \times 465 \\
\hline
159{,}030 & = \overline{342 \times 465}
\end{array}
$$

What is known as the Russian Peasants' Method of Multiplication makes it possible to find the product of any two integers provided one can multiply by 2, divide by 2, and add. Here is an example worked by this method.

Multiply 27 by 38.

Write 27 at the head of one column, 38 at the head of another. In one column — it makes no difference which — multiply by 2, and in the other divide by 2, throwing away ½ if the dividend is odd. The last quotient will always be 1, and the last product will correspond to it. The work below will make this clear.

$$
\begin{array}{r r}
\not{27} & 38 \\
54 & 19 \\
108 & 9 \\
\not{216} & 4 \\
\not{432} & 2 \\
864 & 1 \\
\hline
1026 &
\end{array}
$$

Cross out in the first column every number that stands opposite an even number in the second column, as indicated, and add the numbers which remain. This sum is the desired product, as you can readily show.

Here is another one that is a little longer.

Multiply 47 by 89.

47	89
~~94~~	44
~~188~~	22
376	11
752	5
~~1504~~	2
3008	1
4183	

At first this seems very mysterious, but if you look carefully at the second column, you will see that we did something like it in Chapter II. Let us take it by itself.

		Remainder
2	38	
2	19	0
2	9	1
2	4	1
2	2	0
	1	0

This means that 38 in the dyadic notation is 100,110, that is, $2^5 + 2^2 + 2$.

Now look at the first column

$$1 \times 27 = 27$$
$$2 \times 27 = 54$$
$$2^2 \times 27 = 108$$
$$2^3 \times 27 = 216$$
$$2^4 \times 27 = 432$$
$$2^5 \times 27 = 864$$

This makes it clear that what we have done is to multiply 27 by $2^5 + 2^2 + 2$, that is, by 38. Some terms were omitted because they do not appear in the expression for 38, *i.e.* 2^4, 2^3, and the last term. This accounts for the numbers which we crossed out in the first column.

You may like to try the other example too by this method, expressing 89 in powers of 2. You will find that 47 was multiplied by this very number.

Probably the peasants who multiply by this method have never heard of the dyadic system, and have no idea why it works, but any process is more interesting if we know the reasons which underlie it.

There is a puzzle or trick that is so closely connected with this work that I mention it here. A table like the one shown here is easily prepared. Ask anyone to choose a number in this table, and to tell in which columns it occurs. Then you can tell him what number he chose by adding the topmost numbers in the columns in which the number appears. Suppose, for instance, that the number is found in the first, third, and fifth columns. The numbers to be added are then $1 + 4 + 16 = 21$, which must be the required number.

You will find that each number occurs according to its form when written as a dyad. Thus, $21 = 2^4 + 2^2 + 1$, and so it does not occur in the second and fourth columns.

I	II	III	IV	V
1	2	4	8	16
3	3	5	9	17
5	6	6	10	18
7	7	7	11	19
9	10	12	12	20
11	11	13	13	21
13	14	14	14	22
15	15	15	15	23
17	18	20	24	24
19	19	21	25	25
21	22	22	26	26
23	23	23	27	27
25	26	28	28	28
27	27	29	29	29
29	30	30	30	30
31	31	31	31	31

If you would like a smaller table with only four columns, each will end in 15, as you can readily show.

Or you can, if you wish, make this trick more elaborate by putting in a sixth column. This will begin with 32, and contain the next 31 integers, the last number in the column being 63. This means that all the other columns must be lengthened according to the rule followed in writing them, which you will not find it hard to discover. For instance, the column that begins with 4 contains four consecutive numbers and then skips four. Finally each column will end in 63 instead of 31.

If the druggist in the problem on page 21 had had this diagram hanging near his scales, he could have told easily just what weights to use to add up to any desired number of ounces. Do you see how he could use it?

A diagram may also be made to help the druggist who puts his weights in either or both sides of the scales. Putting a weight with the thing to be weighed amounts to subtracting it from the weight on the other end. That is, each weight may be used in either of two ways, as positive or negative, depending upon where it is put. Weights which stand in the negative column go with the thing that is being weighed, those in the positive column go in the other side of the scales.

In this diagram 7 stands in the columns headed 9, -3, and 1, and $9 - 3 + 1 = 7$. This means that, if a 3-ounce weight with the article to be weighed balances a 9-ounce and a 1-ounce weight at the other end, the article must weigh 7 ounces. Any number of ounces up to 40 may be read from the table in the same way. The table is less regular in form than the one to the base 2.

The principle used here makes it possible to write any number to the base 3 without using 2 at all, but introducing negative terms. For example, instead of writing 8 as $2 \times 3 + 2$, we may write $(3^2 - 3) + (3 - 1)$, substituting for 2, wherever it occurs, $3 - 1$.

I	II	III	IV	V	VI	VII
(1)	(– 1)	(3)	(– 3)	(9)	(– 9)	(27)
1	2	2	5	5	14	14
4	5	3	6	6	15	15
7	8	4	7	7	16	16
10	11	11	14	8	17	17
13	14	12	15	9	18	18
16	17	13	16	10	19	19
19	20	20	23	11	20	20
22	23	21	24	12	21	21
25	26	22	25	13	22	22
28	29	29	32	32		23
31	32	30	33	33		24
34	35	31	34	34		25
37	38	38		35		26
40		39		36		27
		40		37		28
				38		29
				39		30
				40		31
						32
						33
						34
						35
						36
						37
						38
						39
						40

We will illustrate this, taking a larger number than those in this table.

$$74 = 2 \times 3^3 + 2 \times 3^2 + 2,$$
$$= (3 - 1)3^3 + (3 - 1)3^2 + 3 - 1,$$
$$= 3^4 - 3^3 + 3^3 - 3^2 + 3 - 1,$$
$$= 3^4 - 3^2 + 3 - 1.$$

Any number may be written to the base 3 in the usual way, and then, if desired, changed to this form by substituting $3 - 1$ for every 2.

This method of writing makes it possible to carry out the process of multiplication when one knows how to multiply by 3, divide by 3, and add. The work is similar to that when 2 was used, and if the remainder after dividing by 3 is 1, it is disregarded as in the former case. But if the remainder is 2, the quotient is increased by 1, and the remainder is called -1. It makes no difference whether we say $14 \div 3 = 4$ with a remainder of 2, or $14 \div 3 = 5$ with a remainder of -1. In other words, $\frac{14}{3} = 4 + \frac{2}{3} = 5 - \frac{1}{3}$.

So now let us multiply 23 by 46. Except for the difference already noted, we begin just as we did when working with 2.

		Remainders	
$1 \times 23 = 23$	46	$+1$	
$3 \times 23 = 69$	15	0	
$3^2 \times 23 = 207$	5	-1	$46 = 3^4 - 3^3 - 3^2 + 1.$
$3^3 \times 23 = 621$	2	-1	
$3^4 \times 23 = 1863$	1	$+1$	

You can see now what must be done. Add the numbers in the column headed 23 which stand opposite $+1$, subtract from this result the sum of those which stand opposite -1. Those numbers which stand opposite 0 are to be omitted.

Here is one more product which we found by using 2 as a base.

Multiply 47 by 89.

47	89	− 1
141	30	0
423	10	+ 1
1269	3	0
3807	1	+ 1
4183		

It is evident that 2 and 3 are the only numbers that can be used conveniently in this special kind of multiplication, since in division by 4, for instance, there may be any one of three remainders, while in using 2 there is only + 1, and in using 3 only + 1 and − 1.

CHAPTER IV

MOSTLY ON SQUARES

Many centuries ago a man who was famous as mathematician, philosopher, and traveler made a discovery which seemed to him so wonderful that we are told he sacrificed a hundred oxen to celebrate it. But if Pythagoras had been able to look down the centuries and see the millions who were to know and use his theorem, he might have had an even greater celebration. For few statements are oftener repeated and used, even in quite advanced mathematics, than "In any right triangle the square on the hypotenuse is equal to the sum of the squares on the other two sides." You may be interested to know that more than one hundred proofs of this theorem have been discovered. Indeed, it is quite possible that some of you may discover brand new ways to prove it.

But what I wish to speak of now is not the number of proofs of this theorem, but the number of triangles of a certain kind to which it can be applied.

The most familiar numbers which can measure the sides of a right triangle are 3, 4, 5. Of course any multiple of these will also answer. Thus, if a triangle is drawn with sides 6, 8, 10, it will be a right triangle. But this triangle has the same shape as the other, and it is quite possible for two draftsmen to choose scales so that a triangle with sides 3, 4, 5, drawn to one scale, shall be exactly the same size as one with sides 6, 8, 10, drawn to the other scale. But suppose we ask how many right triangles of different shapes can be drawn with sides all represented by integers. We may know already of the combination 5, 12, 13, or 7, 24, 25. Is the number limited, or is there an endless supply of such sets of numbers? And is there a means by which they may be found? Of course we know that, when we are looking for a general answer to such a question, nothing gives quite so simple and usable an answer as Algebra. A formula has been found which gives a general value of three sides of a right triangle. The three forms $m^2 - n^2$, $2\,mn$, and $m^2 + n^2$ are readily shown to represent numbers which measure the sides of a right triangle, since

$$(m^2 - n^2)^2 + (2\,mn)^2 = (m^2 + n^2)^2.$$

If integral values are substituted for m and n, an endless series of solutions of the type which we are seeking will be given. For instance, let $m = 2$, and $n = 1$, the sides are 3, 4, 5; let $m = 3$, $n = 2$, the sides are 5, 12, 13. In order that different shaped triangles may in each case be found, m and n should not both be odd nor both even. (What will happen if they are?)

Perhaps it is a surprise to you to find that there is no end to the number of integers which satisfy the equation $x^2 + y^2 = z^2$. It might interest you to see how many right triangles you can find with sides measured in integers, no side being greater than 100. That will be a nice little problem to investigate.

And now for some further discussion of squares. In all this chapter it will be assumed that we are dealing with integers only. You may, by the way, be interested to learn that a quite difficult branch of Mathematics, called Theory of Numbers, is a study of integers only. Perhaps you have not realized that so much can be done with these simple quantities, 1, 2, 3, 4, ...

Can you tell at a glance whether a given number can possibly be a square? For example, which, if any, of the following numbers might possibly be a square?

3127, 1635, 4852, 748, 923.

Only a glance is needed to show that not one is a square. And why? The reason is evident if you will notice the last figure of the square of each number below 10.

Number	1	2	3	4	5	6	7	8	9
Last figure of square	1	4	9	6	5	6	9	4	1

This shows that no square can end in 2, 3, 7, or 8. But how about 5? Take any number ending in 5, square it, and you will find that it ends in 25. In general, $10a + 5$ represents any number ending in 5.

$$(10a + 5)^2 = 100a^2 + 100a + 25.$$

The coefficient 100 in the first and second terms means that the last two digits are 2 and 5. So then, unless a number which ends in 5 ends in 25, it has no chance to be a square.

Note how the numbers in the above table run: the square of any number c ends in the same digit as the square of $10 - c$. Why?

Try a table of cubes, like the one for squares, and see whether it is easier to detect a non-square that has accidentally strayed into the company of squares or a non-cube similarly out of place.

There is another test of squares which is interesting. If the digits of any square are added, and, in case the sum is greater than 10, the digits of the sum are added, and so on, until a sum less than 10 is reached, this final sum must be 1 or 4 or 7 or 9.

Take, for example, these three numbers and add their digits.

Number	Sum	Second Sum	Third Sum
8,573,984	44	8	
9,545,079	39	12	3
5,975,281	37	10	1

The first two numbers cannot be squares. The last result simply means that nothing against the squareness of this number has as yet come to light. As a matter of fact it is not a square.

The principle involved in this test is treated in the Theory of Numbers under the somewhat imposing title of Quadratic Residues. Like many pompous folks, this is much more ordinary than it looks, for it means the remainders that are left when squares are divided by some given number. For example, what are the quadratic residues of 3? These are found by dividing 1, 4, 9, 16, · · · by 3. You will think it odd that 2 is never left over when a square is divided by 3, no matter how many you try. Every square is either divisible by 3 or leaves a remainder of 1. The quadratic residues of 7 are 1, 4, 2, and there is no difficulty in making out the list for any given number.

Take the squares in order and find the quadratic residues of several numbers. You will see some interesting arrangements; for example, using 7, the remainders are

1 4 2 2 4 1 0 1 4 2 2 4 1 0 . . . .

The reason why these numbers repeat you could easily find for yourselves.

Now the quadratic residues of 9 are found to be 1, 4, 7, and no others (except 0, which, although a remainder, is not counted among the residues). These numbers, if we remember that a remainder of 9 in dividing by 9 really means no remainder at all, give just the set of numbers which I gave you for testing the squareness of a number by adding its units — 1, 4, 7, 9.

This connection may seem a little hard to see, but here are the reasons in an orderly arrangement:

1. If any square is divided by 9, the only possible remainders are 1, 4, 7, 9 (or 0).

2. If any number is divided by 9, and the remainder is not 1, 4, 7, 9 (or 0), that number is not a square.

3. The easiest way to find the remainder when a number is divided by 9 is to add the digits, and divide that sum by 9.

4. Therefore, unless the final sum of the digits is 1, 4, 7, or 9, the number is not a square.

Notice that we have proved in the course of this chapter that, when a square is divided by 3, the remainder is never 2, so that a number that is just 2 greater than a multiple of 3 cannot be a square. This is really very easy to see, but notice what airs it puts on when we dress it up a bit — "Prove that a number of the form $3n + 2$ cannot be a square." Plenty of college seniors would have a hard time proving it, if it were sprung upon them without any introduction, but you see that, with some facts to pave the way, it is quite simple.

Another way to find the quadratic residues of a number is to use a principle found in Chapter I. For example, every number falls under one of the three heads, $3n$, $3n + 1$, $3n - 1$. The square of $3n$ is, of course, divisible by 3. The square of each of the other numbers gives two terms which are divisible by 3 and a remainder of 1. But suppose that, instead of $3n - 1$, we use $3n + 2$, how does that come out? We can use this same method for any number, and it shows clearly why any number c never has more than $\frac{c}{2}$ quadratic residues.

•

Now let us look at squares from still another point of view. Did you ever notice what happens when you add consecutive odd numbers, beginning with 1?

$$1 + 3 = 4,\ 1 + 3 + 5 = 9,\ 1 + 3 + 5 + 7 = 16, \cdots.$$

These sums are all squares, and the sum of two terms gives 2^2, of three terms 3^2, etc. Will the sum of the first ten odd numbers be 10^2? And will this rule hold no matter how far out we go?

Again Algebra gives a satisfactory answer to our question. If you have studied the subject of Progressions, you know that

$$1 + 3 + 5 + \cdots + 19 = \frac{1 + 19}{2} \times 10 = 10^2,$$

or, in general,

$$1 + 3 + 5 + \cdots + (2n-1) = \frac{(2n-1)+1}{2} \times n = n^2.$$

Perhaps some of you have not studied Progressions yet, or, even if you have, you may like to see a different way to prove this formula. Another method is given in Chapter VI.

The table of cubes which I suggested earlier shows that it is not an easy matter to decide whether a given number is a cube. Suppose you continue the table farther. Make a list of the nine digits, then under each write the last digit of its square, under these the last digit of its cube, etc. See what it tells you about fourth and fifth powers.

Nine digits	1	2	3	4	5	6	7	8	9
Second power ends in	1	4	9	6	5	6	9	4	1
Third power ends in .	1	8	7	4	5	6	3	2	9
Fourth power ends in	1	6	1	6	5	6	.	.	.
Fifth power ends in .	1	2	.	.	.	.	.	.	.

Here is still another question about squares. We have found that there is an endless number of integers that satisfy the equation $x^2 + y^2 = z^2$. If we select some special value for x, are there always corresponding values for y and z? It is readily seen that, if $x = 2$, there is no corresponding pair of values for y and z. For if $4 + y^2 = z^2$, there are two squares which differ by 4. Now 4 and 1 differ by 3, 9 and 4 by 5, and any other two differ by more than 5. It follows that $z^2 - y^2$ cannot be equal to 4.

But every odd number except 1 is a possible value for x, for

$$3^2 + 4^2 = 5^2,$$
$$5^2 + 12^2 = 13^2,$$
$$7^2 + 24^2 = 25^2,$$
$$9^2 + 40^2 = 41^2,$$
$$11^2 + 60^2 = 61^2,$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot$$

Do you see a law by which these numbers may be found? $3^2 = 9 = 4 + 5$, $5^2 = 25 = 12 + 13$, etc. This seems to show that, if one squares the odd numbers, and divides the square into two parts differing by 1, these two form the values of y and z. Let us see whether this is in general true.

Any odd number, $2n+1$, squared gives $4n^2+4n+1$. This may be broken up into $2n^2+2n$ and $2n^2+2n+1$, and these three numbers satisfy the equation $x^2+y^2=z^2$, since

$$(2n + 1)^2 + (2n^2 + 2n)^2 = (2n^2 + 2n + 1)^2.$$

In the above solutions, giving odd values to x, y is always even. We may obtain even values for x also by multiplying any of the above results by 2^2. From the equation $3^2 + 4^2 = 5^2$ we thus derive the equation $6^2 + 8^2 = 10^2$, and so in general any even number except 2 may be substituted for x or y.

For z the case is different, its range of values is more limited. It is never divisible by 3 or 7 or 11, for instance, unless x and y are each divisible by that number. You can prove this yourselves, with a hint to guide you.

To prove that, if x and y are not both divisible by 3, z cannot be divisible by 3.

If x is not divisible by 3, it must have one of the two forms $3n + 1$ or $3n - 1$. Similarly y must be of the form $3k$ or $3k + 1$ or $3k - 1$. (Do you see why we use both n and k?) Now square and add, using all possible combinations of signs in x and y, and see whether 3 will divide any one of the sums which you get. You can try other numbers in the same way.

We see then that some numbers cannot be written as the sum of two squares. It is, however, an interesting fact that every number can be written as a single square or as the sum of two or three or four squares. More than four are never needed for any number. Often a number can be written as the sum of squares in several different ways. But you will notice as you experiment that a multiple of 3 or 7 or 11 is never the sum of just two squares unless these two squares each contain that same factor. Here are several illustrations:

$1 = 1,$
$2 = 1 + 1,$
$3 = 1 + 1 + 1,$
$4 = 4 = 1 + 1 + 1 + 1,$
$9 = 4 + 4 + 1,$
$12 = 4 + 4 + 4 = 9 + 1 + 1 + 1,$
$28 = 25 + 1 + 1 + 1 = 16 + 4 + 4 + 4 = 9 + 9 + 9 + 1,$
$55 = 49 + 4 + 1 + 1 = 25 + 25 + 4 + 1 = 36 + 9 + 9 + 1,$
$225 = 81 + 144$. A multiple of 3 is the sum of only two squares. Why?

PROBLEMS

1. Show that by arranging the digits 1, 6, 9 in different orders three different squares may be obtained.

2. Show that by arranging the digits 1, 4, 6, 7 in different orders two different squares may be obtained.

3. If $x^2 + y^2 = z^2$, and x, y, and z are integers, prove that either x or y is a multiple of 3; also either x or y is a multiple of 4; also x or y or z is a multiple of 5.

4. Prove that the following method of squaring a number which ends in 5 is correct. Take the number left when 5 is cut off, and multiply it by the next larger number. To this product annex 25. For example, to square 35 take 3 and multiply it by 4, annex 25. The result is 1225. To prove this method correct in general square $10a + 5$, and factor the first two terms.

5. Prove that if a number ends in 5, its square has an even digit just before the final 25.

6. Prove that a number that ends in 425 or 825 cannot be a square. You will find the reason in example 4.

7. Prove that if a square ends in 6, the digit just before 6 is always odd; if it ends in any other number, the next to the last digit is always even.

8. Show that it is impossible to find two consecutive odd integers the sum of whose squares is the same as the sum of the squares of two consecutive even integers.

9. Prove that the difference of two squares is never a prime number, except that $3 = 2^2 - 1^2$.

10. Show that every square has one of the two forms $4n$ or $8n + 1$.

11. Show that no square can be made up entirely of odd digits.

12. All squares made up entirely of even digits, and not ending in zero (which is generally included among the even digits), end in the same digit. What is it?

13. Prove that the sum of two squares is never divisible by 7 or 11 unless each square is so divisible. Is the same kind of statement true of 13 or 17?

14. Find a square made up of four digits, of which the first and second are the same, and the third and fourth are the same; that is, it has the form *aabb*.

Suggestion. Find a reason why 4 is the only possible value for b. For instance, if $b = 1$, the square ends in 11, which is impossible. Why?

15. Find a square whose digits in order are c, $c - 2$, $c - 2$, $c + 1$.

Suggestion. The final digit, $c + 1$, must be 4, 5, 6, or 9, so that c must be 3, 4, 5, or 8. Try 3. The resulting number, 3114, is not a square. Why? Try 4, and you will obtain 4225, which is a square. The next problems are solved in a similar way. Note that, if a perfect square is divisible by 2, or 3, or any prime number, it must contain that factor a second time.

16. The digits of a square in order are k, $k + 1$, $k + 2$, $3k$, $k + 3$. Find the square.

17. The digits of a square in order are t, $t + 1$, $t + 3$, $t + 5$, $t + 2$. Find the square.

18. Find a square whose digits in order are d, $d + 1$, $d + 1$, $d + 2$, $d - 2$, $d + 1$.

19. Find two numbers which have the same digits but in opposite order, and their squares have the same property. For example, 12 and 21 have as squares 144 and 441. Find other instances.

20. Find a square which consists of four digits, and when the digits are written in opposite order this new number is also a square.

Suggestion. Show that the sum of these two numbers must be divisible by 11, then use Example 13.

21. Use Examples 19 and 20 on page 12 to make up a test that can be used to tell whether a given number has a chance of being a perfect cube.

CHAPTER V

THE CHARM OF DECIMALS

It is probable that when you were learning to turn fractions into decimals you thought it monotonous and uninteresting. In fact very few people enjoy dividing unless their attention is called to some interesting sights along the way, and some of those sights I hope now to show you. We shall take it for granted that the fractions which we use are in their lowest terms.

You know that when you change fractions into decimals, the division sometimes comes to an end, and sometimes can be continued indefinitely. For example,

$$\tfrac{1}{4} = .25, \quad \tfrac{1}{20} = .05, \quad \tfrac{1}{3} = .333 \cdots.$$

In the first two cases the division finally comes out exact, in the last case it never will, however long the process is continued. It is clear that when the denominator is a factor of 10, or 100, or 1000, etc., there will finally be no remainder, but in every other case there will always be a remainder. This means that the division comes to an end only when the denominator has as factors no prime numbers except 2 and 5.

$$\frac{1}{2^3 \times 5^2} = \frac{5}{2^3 \times 5^3} = \frac{5}{10^3} = \frac{5}{1000} = .005.$$

This shows that, if the denominator is a product of twos and fives, it can be turned into powers of 10 by multiplying numerator and denominator by some power of 5 or of 2. This is the only case in which a fraction can be represented by a decimal with a definite number of digits. In all other cases, that is, when the denominator contains a factor which is not divisible by 2 or 5, division may be carried on indefinitely. This may at first seem rather discouraging, but it is not really so bad as it sounds, for after a time the digits begin to repeat, and every such fraction leads to what is called a circulating or repeating decimal. Lewis Carroll once said that a department of Mathematics ought to own a lantern to exhibit circulating decimals in the act of circulating. That is what I shall try to do now, show how these decimals circulate.

In a circulating decimal the numbers which repeat are called the period of the decimal. In $\frac{1}{3}$ the period is 3, in $\frac{1}{27}$ the period is .037. This is often indicated by drawing a line above the period, for instance, $\frac{1}{3} = .333 \cdots = .\overline{3}$, $\frac{1}{27} = .037037 \cdots = .\overline{037}$. Why is it that in such forms the digits begin to repeat after a time? The answer is that they can't help themselves. Suppose that you are changing $\frac{1}{7}$ to a decimal. The only possible remainders as you divide are 1, 2, 3, 4, 5, 6, and so, after at the most six divisions, one of the earlier remainders must appear, and so the numbers in the quotient are repeated. Perhaps this will be clearer if we actually do this by long division, noting the remainders.

```
7)1.000 · · · (.142857
  7
  30
  28
   20
   14
    60
    56
     40
     35
      50
      49
       1
```


Beginning with 1, the successive remainders are 3, 2, 6, 4, 5, and then 1 appears again, and so the six figures in the dividend are repeated. Whenever the division does not come to an end, leaving no remainder, this sort of thing is sure to happen; the numbers will begin to repeat. If you are changing $\frac{1}{11}$ to a decimal, there cannot be more than 10 figures before the numbers already found begin to repeat, and in the case of $\frac{1}{13}$ there cannot be more than 12 digits in the period. You will find by experimenting that, while the number of digits cannot be greater than a certain fixed number, it may be much less. For example, there are two possible remainders when one divides by 3, but $\frac{1}{3}$ gives $.\overline{3}$, only one digit in the period. In dividing by 9 there are 8 possible remainders, but $\frac{1}{9} = .\overline{1}$, only one digit in the period, $\frac{1}{11}$ has only 2 digits in its period, while $\frac{1}{13}$ has 6. In fact you will find that 7 is the first denominator to give a period of maximum length, and that the next one is 17; that is, if $\frac{1}{17}$ is expressed as a decimal, there are 16 digits before repetition begins.

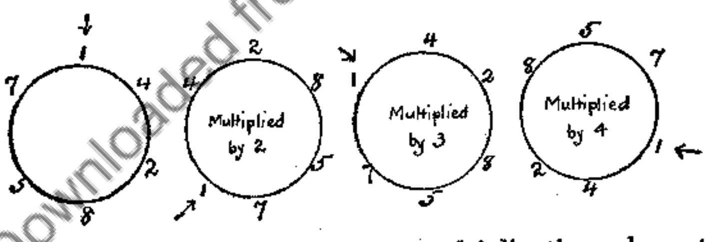
Now I shall show you some interesting facts about the period to which $\frac{1}{7}$ leads. I am using chiefly fractions with 1 as numerator for the same reason that in Geometry or Trigonometry we study triangles more than polygons — the harder forms can be studied readily through the simpler forms.

$$\frac{1}{7} = .\overline{142857}.$$

Multiply the period by 2.

$$2 \times 142,857 = 285,714.$$

The figures are just the same but rearranged, and you notice that they are not badly mixed. Suppose we put them around a circle in clockwise order, and multiply by 2, 3, 4, 5, 6, beginning at the end, that is, with 7, each time.



You see at once that these multiplications do not change what we call the "cyclic order," that is, the order on the circle; and if these circles were wheels that we could turn around, each multiplication would simply turn the wheel a little.

When you try multiplying by 7, something different happens, as you see must be the case.

You may like to experiment further, trying 8, 9, etc., but in this case you notice that there is something to carry from the following period, which is not the case with smaller numbers.

Another interesting result is obtained by adding .99999 · · · to the period.

$$\begin{array}{r} .142857142857\cdots \\ \underline{.999999999999}\cdots \end{array}$$

As you begin to add note that there is 1 to carry.

Now we will try another experiment with the period 142,857. Divide it into two parts, 142 and 857, and add the first digits of each, then the second digits of each, and then the third.

$$1 + 8 = 4 + 5 = 2 + 7 = 9.$$

Take the sum of all the digits,

$$1 + 4 + 2 + 8 + 5 + 7 = 27.$$

Now arrange the period in sets of two digits each and add.

$$14 + 28 + 57 = 99.$$

Or, taking a different cyclic order,

$$42 + 85 + 71 = 198.$$

And finally let us add the two 3-digit numbers which we get by separating the period into halves,

$$142 + 857 = 999$$
$$428 + 571 = 999$$
$$285 + 714 = 999$$

You notice that every one of these sums is divisible by 9.

If you change $\frac{1}{13}$ to a decimal, you will find its period to be 076923, and you get exactly the same kind of results when you combine the digits in the same way. Isn't this remarkable?

What we said in the former case about multiplying by 2, 3, 4, 5, 6, does not hold exactly for $\frac{1}{13}$ because its period is only half as long as it might be expected to be. You will find that, when you multiply by 1, 3, 4, 9, 10, 12, you get the same digits in cyclic order, but, if you multiply by 2, 5, 6, 7, 8, 11, you get a new period, 153,846, from which you can get the same results by addition as before.

Add the digits singly,	$1 + 5 + 3 + 8 + 4 + 6 =$
Add them by twos,	$15 + 38 + 46 =$
	$53 + 84 + 61 =$
Add them by threes,	$153 + 846 =$
	$538 + 461 =$
	$384 + 615 =$

Notice a result that you can get from each of those sets of multipliers:

1	3	4	9	10	12
2	5	6	7	8	11

The two numbers at the ends add up to 13, so do those next to the ends, and the two middle ones.

You will find the same kind of results for other repeating decimals when the denominator is a prime number; $\frac{1}{17}$, for instance, has 16 digits in its period, and you can make the same kind of combinations with them. If the denominator is larger than 10, the period must begin with one or more zeros. When you have half the digits, it is easy to write the other half without further division, owing to the fact that we found in the period of 7, and saw to be true also in the short period of 13, that, if the period is cut in halves, the sum of two corresponding digits in each half is 9. So, if you know the first eight digits of the period of 17, you can write the other eight. Here are the first ten:

0588235294.

You can fill in the other six, and try the same experiments which we tried with the period of 7. Arrange these digits around a circle, and multiply by 2, 3, 4, · · · 16, always counting 0 as the first figure. You will find that the product is already there on the circle, but it is revolved into a different position.

Also, if you add the digits by ones, twos, fours, eights — all possible combinations that divide them into groups containing the same number of digits in each — you will find that each sum is a multiple of 9.

$5 + 8 + 8 + 2 + 3 +$
$58 + 82 + 23 + 52 +$
$5 + 88 + 23 + 52 +$
$588 + 2352 +$
$5882 + 3529 +$
etc.

Not only the quotients but the remainders, too, have special properties. We found, page 46, that when we divide by 7, the remainders are 1, 3, 2, 6, 4, 5. Divide these into two groups, and take the sum of the corresponding digits in each.

$$132645; \quad 1 + 6 = 3 + 4 = 2 + 5 = 7.$$

Or, adding every other one,

$$1 + 2 + 4 = 7, \quad 3 + 6 + 5 = 14.$$

The sum of all six is

$$1 + 3 + 2 + 6 + 4 + 5 = 21.$$

Take them by twos,

$$13 + 26 + 45 = 84, \quad 32 + 64 + 51 = 147.$$

Take them by threes,

$$132 + 645 = 326 + 451 = 264 + 513 = 777.$$

It is quite evident that these remainders have something to do with 7.

You will find the six remainders when $\frac{1}{13}$ is changed to a decimal to be 10, 9, 12, 3, 4, 1. These can be treated in the same way as the remainders for 7, but now the number which keeps appearing is 13.

When the remainders are not all single digits, it is a little harder to handle them, and so I will write out some of these combinations. You see that when two single digits like 3 and 4 come together, we use 34 as one of the numbers to be added, that is, we really use 30 + 4. So now, when 10 and 9 come together, we must write $100 + 9$, that is, 109, and when 9 and 12 come together, we use $90 + 12$, that is, 102. So, though it looks different, it is really done in the same way.

$$109 + 123 + 41 =$$
$$102 + 34 + 20 =$$

Notice how many times each of these sums contains 13 as a factor, and compare this result with the corresponding forms for 7.

$$1102 + \cdots =$$
$$\cdots + 420 =$$
$$\cdots + 209 =$$

You will enjoy filling out the blanks, and finding how the sums are related to 13.

Since $\frac{1}{17}$ leads to a sixteen-figure period, there are wonderful chances to make combinations with its remainders; each sum obtained by the same methods as in these two cases will have 17 as a divisor.

Now we ask how it is possible to find from a given repeating decimal the fraction which gave rise to it. For example, suppose the given decimal is $.777 \cdots$.

Let $\quad x = .7777 \cdots$

Then, $\quad 10x = 7.7777 \cdots$

$9x = 7$, the decimals being exactly alike.

$x = \frac{7}{9}$.

To reduce .3535 · · · to a fraction.

$$\begin{aligned} \text{Let} \quad x &= .3535\cdots \\ 100\,x &= 35.3535\cdots \\ \hline 99\,x &= 35 \\ x &= \tfrac{35}{99}. \end{aligned}$$

To reduce .370370 · · · to a fraction.

$$\begin{aligned} \text{Let} \quad x &= .370370\cdots \\ 1000\,x &= 370.370370\cdots \\ 999\,x &= 370 \\ x &= \frac{370}{999} = \frac{370}{9\times 111} = \frac{370}{9\times 3\times 37} = \frac{10}{27}. \end{aligned}$$

From these examples we may deduce a rule for changing a repeating decimal to a fraction. Take the period for the numerator, and for the denominator take as many nines as there are digits in the period.

This result may also be obtained by using a geometric progression. For example, the repeating decimal .777 · · · is equal to the progression

$$\tfrac{7}{10} + \tfrac{7}{100} + \tfrac{7}{1000} + \cdots.$$

Here $a = \frac{7}{10}$, $r = \frac{1}{10}$, and n is indefinitely large. In this case the formula becomes $S = \dfrac{a}{1-r} = \dfrac{7}{9}$.

In the second illustration,

$$\overline{.35} = \frac{35}{100} + \frac{35}{100^2} + \frac{35}{100^3} + \cdots.$$

Now substitute the values of a and r, and find the value of S.

In case the repeating decimal is preceded by a whole number, as in 2.777 · · · , the repeating decimal may be used alone, and the fraction found from it may be added to the whole number; thus, 2.777 · · · = $2\frac{7}{9}$. But the method which we used before will work.

$$\begin{aligned} x &= 2.777 \cdots \\ 10\,x &= 27.777 \cdots \\ \hline 9\,x &= 25 \\ x &= 2\tfrac{7}{9}. \end{aligned}$$

If there are digits immediately after the decimal point before the repeating decimal begins, the same method will work. For example, let us reduce to a fraction .85323232 · · ·.

$$\begin{aligned} x &= .853232 \cdots \\ 100\,x &= 85.323232 \cdots \\ 10000\,x &= 8532.323232 \cdots \\ \hline 9900\,x &= 8447 \\ x &= \tfrac{8447}{9900}. \end{aligned}$$

We now have enough material on hand to try to answer the question, why do some numbers give a repeating decimal with the largest possible number of digits in the period, while others have a shorter period?

If we know the number of digits in a given period, we have shown that we must use the same number of 9's in the denominator. And so we see that if in some way we could find the number of 9's in the denominator, we should know just how long the period is.

Let us, for example, investigate the period of $\frac{1}{11}$. Suppose x stands for this period. Then

$$\frac{1}{11} = \frac{x}{9999 \cdots},$$

the number of 9's in the denominator being exactly the same as the number of digits in x. If we divide the denominator on each side of the equation by 11, the left side becomes 1, and therefore the right side must be 1. Now, how many 9's are needed in order that the right-hand denominator may be divisible by 11? Clearly only two. Therefore, $x = \frac{99}{11} = 09$.

Let us try $\frac{1}{7}$ in the same way.

$$\frac{1}{7} = \frac{x}{99999 \cdots}.$$

Now see how many 9's are needed in order that the second denominator may be divisible by 7. You will find that six are necessary, and so the period of $\frac{1}{7}$ is six digits long.

$$\frac{1}{7} = \frac{x}{999999}.$$

Factor out 7 from each denominator,

$$1 = \frac{x}{142857}.$$

Therefore $x = 142{,}857$, our old friend.

This makes it easy to see why $\frac{1}{3}$ and $\frac{1}{9}$ have each only one digit in the period. It happens that both 7 and 13 will divide 999,999, so that they each have a six-digit period.

$$\begin{aligned} 999{,}999 &= 9 \times 111{,}111 = 9 \times 3 \times 37{,}037 \\ &= 9 \times 3 \times 37 \times 1001 \\ &= 3^3 \times 7 \times 11 \times 13 \times 37. \end{aligned}$$

You can see that this looks as if 37 also would have a six-digit period, but it happens that it does not require so many 9's; it will divide 999, and so 37 has only a three-digit period: $\frac{1}{37} = .\overline{027}$.

Instead of using 9, 99, 999, etc., these numbers are usually written in the special form $10 - 1$, $10^2 - 1$, $10^3 - 1$, etc., so that the rule may be stated in this way:

The length of the period of the fraction $\frac{1}{n}$, when n is a prime number, is determined by the lowest value of k for which $10^k - 1$ is divisible by n.

The following table of prime factors of $10^k - 1$, when $k = 1, 2, 3, \cdots$ may be of interest.

$$
\begin{aligned}
10^1 - 1 &= 3^2 \\
10^2 - 1 &= 3^2 \times \underline{11} \\
10^3 - 1 &= 3^3 \times \underline{37} \\
10^4 - 1 &= 3^2 \times 11 \times \underline{101} \\
10^5 - 1 &= 3^2 \times \underline{41} \times \underline{271} \\
10^6 - 1 &= 3^3 \times \underline{7} \times 11 \times \underline{13} \times 37 \\
10^7 - 1 &= 3^2 \times \underline{239} \times \underline{4649} \\
10^8 - 1 &= 3^2 \times 11 \times \underline{73} \times 101 \times \underline{137} \\
10^9 - 1 &= 3^4 \times 37 \times \underline{333{,}667} \\
10^{10} - 1 &= 3^2 \times 11 \times 41 \times 271 \times \underline{9091}
\end{aligned}
$$

When a number is underlined, it appears for the first time, and the number of digits in its period is the corresponding exponent of 10.

Notice what short periods some quite large numbers have; 271, for instance, and 333,667. If $\frac{1}{101}$ is expressed as a decimal, instead of 100 digits in its period, as we might expect to find, it has only 4. You may like to try some of these numbers and see that this rule really works.

There are only nine numbers less than 100 that have the full-length period, the last one being 97, so that $\frac{1}{97}$ has 96 digits in its period. Of course this means that $\frac{2}{97}, \frac{3}{97}, \cdots \frac{96}{97}$ have periods of the same length, and each is made up of the same digits in the same cyclic order.

All these statements about the periods and remainders of a decimal are true in general only when the denominator is prime. For example, $\frac{1}{21} = .\overline{047619}$, a period that lacks the special characteristics of 142,857. You will still find in many cases that the sum of the digits is a multiple of 9.

$$\frac{1}{39} = .\overline{025641},\ 2 + 5 + 6 + 4 + 1 = 18$$
$$\frac{1}{77} = .\overline{012987},\ 0 + 9 = 1 + 8 = 2 + 7,$$
$$1 + 2 + 9 + 8 + 7 = 27.$$
$$\frac{1}{231} = .\overline{004329},\ 4 + 3 + 2 + 9 = 18.$$

And now what can we say about decimals which neither come to an end nor repeat after a period of certain length? Of course there are such decimals, and it is easy to write down any number of them. Take, for instance, this decimal, .212112111211112 · · ·, not ending and not repeating. We have shown that a fraction always leads to a decimal which either comes to an end or repeats a certain set of digits, and that, conversely, the fraction which corresponds to either kind of decimal can always be found. So it is clear that a decimal in which the digits follow each other without periodic repetition cannot be a fraction; it must belong to that other kind of numbers which we call irrational.

Take, for instance, $\sqrt{2}$. The root may be extracted by the usual methods, and carried out to as many decimal places as we please. With each new digit that is found we have a number which, when squared, comes a little nearer to 2 than the one just before it. These digits never repeat in regular order; this number has no period. If it had, we could extract the square root of 2 exactly. To seven decimal places $\sqrt{2} = 1.4142136$. Now this doesn't look at all different from some repeating decimals, but we have no doubt whatever that, however far the calculation is carried out, even to hundreds or thousands of decimal places, no periodic repetition of the digits will occur.

PROBLEMS

1. Reduce to fractions the following repeating decimals.

$.\overline{324}$, $.\overline{6}$, $.\overline{63}$, $.\overline{018}$, $3.\overline{370}$, $.25\overline{42}$, $1.5\overline{34}$, $.\overline{657}$, $.\overline{857142}$.

2. The following are the first nine digits of the 18-digit period of $\frac{1}{19}$. Complete the period, and experiment with some combinations of its digits.

052631578

3. Here is the first half of another period. Complete the period and experiment on its digits, and see if you can come to a conclusion as to the fraction from which this period comes if its denominator is prime.

04347826086

4. In reducing a certain fraction to a decimal the remainders are

10, 8, 11, 18, 19, 6, 14, 2, 20, 16, 22, 13, 15, 12,

and eight others. Find the others, and decide what the fraction is if the denominator is prime.

The following examples in division are interesting because such a small amount of information enables us to get a rather long — sometimes a very long — answer. They all have to do with shifting a digit from the beginning to the end of a number, as, for instance, changing 43,926 into 39,264. I will give you some suggestions on one which will show you how simple they really are.

5. The first digit of a number is 2. If it is shifted to the end, the new number is half the original number.

If the result of multiplying the new number by 2 is to give a number whose first digit is 2, the first digit of the new number must be 1. That is, dividing 2 by 1 gives the second digit of the original number, which therefore begins 21 · · ·. If we continue to divide by 2, we see that the third digit must be 0, 210 · · ·, the next one is 5, 2105 · · ·, and you will have little difficulty in getting the rather long result, 210,526,315,789,473,684. If you shift the initial 2 to the end of the number and multiply by 2, you will get this number of 18 digits.

It is just as easy to begin at the other end. The final digit after the shift is made is 2; therefore the final digit of the original number must be 4, since it is twice this new number. For the same reason, the figure before 4 must be 8, etc.

6. The first digit of a number is 4. If it is shifted to the end, the new number is one fourth the original number.

You will find this easier, as there are only six digits in the answer.

7. Try this same example, using, instead of 2 or 4, the digits 3, 5, 6, 7, 8, 9.

There is evidently a close connection between these problems and the repeating decimals which we have been investigating in this chapter. In Ex. 5, for instance, the result contains 18 digits, and it will be found that the sum of the first and tenth, the second and eleventh, etc., is 9. These digits are the same as those that are found in reducing $\frac{4}{19}$ to a decimal. When 3 is used, the digits are the same as those found in reducing $\frac{9}{29}$ to a decimal, and the others in order may be found by using $\frac{16}{39}$, $\frac{25}{49}$, $\frac{36}{59}$, and in general $\frac{x^2}{10x-1}$ when $x = 2, 3, \cdots 9$.

There are many other problems like these with which you can experiment, as the following examples show.

8. Find a number beginning with 6 which gives a number one fourth as large when 6 is shifted to the end.

9. Find a number beginning with 4 which gives a number one half as large when 4 is shifted to the end.

10. Find a number beginning with 7 which gives a number one third as large when 7 is shifted to the end.

11. Find a number beginning with 3 which gives a number one seventh as large when 3 is shifted to the end.

Try other problems of this kind for yourselves, and see how easy it is to make them up. At times you will probably run across very long numbers.

CHAPTER VI

IS THIS FORMULA TRUE?

I am going to show you a quite simple way to find out whether a formula which you may have found to be true for several numbers that you have tried in it really is true; that is, to find out whether it will hold for every number which you may substitute in it.

You probably know that the fact that a certain formula is true when you substitute 2 or 3 for a letter in it does not prove that it is true for 4.

Take, for example, this formula,

$$1^2 + 2^2 + 3^2 + \cdots + n^2 = n^2 + 9\,n - 22,$$

and let $n = 3$. Then

$$1^2 + 2^2 + 3^2 = 3^2 + 9 \times 3 - 22 = 14.$$

Now try letting $n = 4$:

$$1^2 + 2^2 + 3^2 + 4^2 = 4^2 + 9 \times 4 - 22 = 30.$$

In both these cases the statement is true. But now let $n = 5$:

$$1^2 + 2^2 + 3^2 + 4^2 + 5^2 = 5^2 + 9 \times 5 - 22.$$

While one side gives 55, the other gives 48, and so we are quite sure that the formula is not a true one. For, in order to be accepted and used as a formula, it must be true for any numbers which any one may choose to substitute in it. For example, in this well-known formula,

$$(a + b)^2 = a^2 + 2\,ab + b^2,$$

we may choose for a and b any numbers we please, positive or negative, fractional or integral, and the two members of this equation will have the same value.

Now there is a method by which we may prove a given formula to be true or false without any great expenditure of time. In Chapter IV we proved that the sum of the first n odd numbers is always n^2. Here is another way to prove it which has the advantage of being more general than that earlier method.

Let us ask whether it is always true that the sum of the first n odd numbers is n^2; that is, is this formula true?

$$1 + 3 + 5 + \cdots + (2\,n - 1) = n^2.$$

Do you see why we stop with $2\,n - 1$? If this statement is true, then it is just as true when we go one step further, that is, to $2\,n + 1$. If we add $2\,n + 1$ to the first member, we must add it to the second member also.

$$1 + 3 + 5 + \cdots + (2\,n - 1) + (2\,n + 1) = n^2 + 2\,n + 1.$$

But this means that the sum of the first $n + 1$ odd numbers is $(n + 1)^2$, and so, if this formula is true for any particular number of terms, it is true for the next larger number of terms; that is, it is true when we go just one step further. Be sure you get this. But now see what we have here. We know that the formula is true when $n = 3$, because we actually added the first 3 odd numbers and found 3^2. Therefore it must be true when $n = 4$, because we have shown that, if it holds for any number, n, it holds for the next larger number, $n + 1$. If it is true for 4, it must, for the same reason, be true when $n = 5$, and since we can reach any integer by continuing to add 1 to these earlier values, it holds for any number of terms. And so we have established the fact that the sum of the first n odd numbers is always n^2.

This method of proving a formula is called "Mathematical Induction."

See if you can prove by mathematical induction a formula that you have probably proved by another method:

$$1 + 2 + 3 + \cdots + n = \frac{n(n + 1)}{2}.$$

In order to continue the formula one step further add $n + 1$ to each side. Then factor the right side after reducing its terms to a common denominator, and you will see that you have the same formula as before, except that now it goes out to $n + 1$ terms instead of n. So, if you show that this formula holds for $n = 2$, this proof by induction shows that it holds for $n = 3$, and therefore for $n = 4$, etc.

You notice that this method does not enable us to discover formulas, but we can test by it the truth of a relation which has been noticed in several cases, and which therefore seems likely to be true.

Be sure that you do not make the common mistake of thinking that the use of a few special numbers is all that is necessary. Take a *general* number n, in the formula which you are testing, and show that when $n + 1$ is used, the formula still holds. This does not prove that the original formula is true. It means simply that if you can find a special integer for which it is true, it will hold for the next larger integer. For instance, the formula $1 + 3 + 5 + \cdots + (2n - 1) = n^2$ states that the sum of the first n odd numbers is n^2. We showed that, *if* this is true, then the sum of the first $n + 1$ odd numbers is $(n + 1)^2$. After this general step has been taken, but not before, we are ready to use some one particular instance in which the formula holds, generally trying $n = 2$. If it is true for $n = 2$, it has been shown true for $n + 1$, or 3. It is true for $n = 3$, therefore it is true for $n + 1$, or 4, and so on without end.

Some very important formulas can be established by this method. One of the most important is the Binomial Formula when the exponent is a positive integer.

PROBLEMS

Prove the following seven formulas true.

1. $1 + 5 + 9 + 13 + \cdots + (4n - 3) = n(2n - 1)$.

2. $1 + 4 + 7 + \cdots + (3n - 2) = \dfrac{n(3n - 1)}{2}$.

3. $\dfrac{1}{1\times 2} + \dfrac{1}{2\times 3} + \dfrac{1}{3\times 4} + \cdots + \dfrac{1}{n(n+1)} = \dfrac{n}{n+1}$.

4. $1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{1}{6}\, n(n + 1)(2n + 1)$.

5. $1^2+3^2+5^2+\cdots+(2n-1)^2=\frac{1}{3}n(2n-1)(2n+1)$.

6. $1^3+2^3+3^3+\cdots+n^3=\frac{1}{4}[n(n+1)]^2$.

7. $1^3+3^3+5^3+\cdots+(2n-1)^3=n^2(2n^2-1)$.

8. Show that the formula $(n+1)^2=2n^2-n+3$ is true for $n=1$ and $n=2$. Does it therefore follow that it is a correct formula?

9. The formula $2+4+6+\cdots+2n=n^4-10n^3+36n^2-49n+24$ will be found to be true if 1 or 2 or 3 or 4 is substituted for n. Is it therefore correct?

CHAPTER VII

MAGIC SQUARES

Magic squares are made by arranging the first 9, 16, 25, etc. integers in the form of a square, so that the sum of the numbers in each row or horizontal line, in each column or vertical line, and in each diagonal, is exactly the same.

Here is a magic square made up of the first nine integers. It is said to be of the third order.

8	1	6
3	5	7
4	9	2

You will find that each one of the three rows, the three columns, and the two diagonals adds up to 15.

$$8 + 1 + 6 = 3 + 5 + 7 = 4 + 9 + 2 = 8 + 3 + 4$$
$$= 1 + 5 + 9 = 6 + 7 + 2 = 8 + 5 + 2$$
$$= 4 + 5 + 6.$$

Here is a magic square of the fourth order.

16	2	3	13
5	11	10	8
9	7	6	12
4	14	15	1

You will find that in this case, too, the sum of the numbers in each row, column, and diagonal comes out the same.

It is a rather simple matter to write a magic square of any odd order. The rules for those of even order are much harder, and I shall show you how to write one of the fourth order only. First, write the first 16 integers in a square, taking them in their natural order.

1*	2	3	4′
5	6″	7°	8
9	10°	11″	12
13′	14	15	16*

Now let the numbers that are marked in the same way, for instance 1 and 16, change places with each other. You will find that you get the square of the fourth order already given.

In writing a magic square of any odd order the rule is to make the numbers follow each other, as far as possible, in a northeast direction, that is, they move diagonally upward toward the right. Further rules are needed only when this one cannot be carried out.

It is well to arrange in advance an outline of the square, using dots, thus —

. . .
. . .
. . .

Put 1 in the middle of the top row. It is impossible to move from this point in a northeast direction, so one of the extra rules is needed at once. Here it is: When the top row is reached, but not the last space, the next digit goes down to the bottom row, one space to the right.

The same sort of difficulty arises when we reach any point in the last column, and so can go no farther toward the northeast. The rule then is: Go to the first column, but one space upward.

There is just one more question that arises. Suppose the space in which we should normally put the next number is already occupied, or there is no column to the right and no row above. What then? In this case we go to the space directly under the one we have just reached.

If you look over the square of the third order already given, you will see that it obeys these rules, but the direction has to change so often that it is really easier to follow the rules on a larger square. Look at this skeleton of a square of the fifth order, and you will find an illustration of each of these rules.

.	.	1	.	.
.	5	.	.	.
4	6	.	.	.
.	.	.	.	3
.	.	.	2	.

We put 1 in the middle of the top row. Since no northeast motion is possible, 2 falls in the bottom row, one space to the right, 3 follows the regular course, but 4 goes to the first column, one space higher up, 5 follows the regular course, but 1 is already in the space to which 6 would naturally go, so 6 takes the place just below 5, and then all goes on merrily. These rules work perfectly, and it is fun to write out magic squares of the seventh, ninth, or higher orders.

I write this same square once more, putting in a few more numbers, in case you have difficulty at any point, and would like to refer to it.

.	.	1	8	15
.	.	.	.	16
.	.	.	.	.
.	.	.	21	.
.	18	.	.	.

You will probably have no trouble at all in completing the square, or in writing more elaborate ones, but, just to make sure, I give an outline of a square of the seventh order with some of the harder places filled in, so that you can refer to it in case of need.

.	.	.	.	.	.	.
.	.	.	.	.	.	29
.	.	8	.	.	.	.
5	.	.	.	.	.	.
.	15	.	.	.	.	.
21	.	.	.	.	.	.
.	.	.	.	.	11	.

You will find that by this method the last number always comes out in the last row directly underneath 1, and that the main diagonal, as it is called, is never interrupted. Perhaps you will also notice for yourselves what number stands in the center of the square, and a connection between the numbers which stand at the same distance from that central number and on opposite sides from it.

The next place to be used on reaching the edge can be determined easily by writing an extra row above and an extra column to the right of the form on which you are working, and using as before the northeast rule. The numbers thus located are readily shifted to the bottom row and the left-hand column of the original square. In using this method the only rule needed besides the one in regard to northeast motion is the one about the place directly beneath an occupied place. Here are two illustrations:

.	9	2	
8	1	6	8
3	5	7	3
4	9	2	

.	18	25	2	9	
17	24	1	8	15	17
23	5	7	14	16	23
4	6	13	20	22	4
10	12	19	21	3	10
11	18	25	2	9	

How can we find out what the number is which gives the sum of each row, column, and diagonal of a magic square? We know how to find the sum of all the numbers in the square (page 64) and by dividing by the number of rows, that is, by the order of the square, we find the sum of the numbers in each row.

For example, in a square of the third order,

$$1 + 2 + 3 + \cdots + 9 = \frac{9 \times 10}{2} = 45,$$

and

$$45 \div 3 = 15.$$

In a square of the fourth order,

$$1 + 2 + 3 + \cdots + 16 = \frac{16 \times 17}{2} = 136,$$

and

$$136 \div 4 = 34.$$

In general, if a square is of the nth order,

$$1 + 2 + 3 + \cdots + n^2 = \frac{n^2(n^2 + 1)}{2}.$$

This is the sum of all its numbers. If it is divided by n, the result gives the sum of the numbers in any row or column or diagonal of a magic square, that is, $\frac{n(n^2 + 1)}{2}$. So you can check the correctness of any square very easily.

There are many arrangements of a magic square of any given order. I have given you the method that I like best for constructing squares of odd orders, but there are many other methods, one, for instance, that is based on the knight's move in chess.

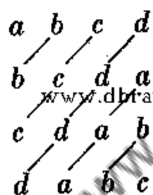
The square of the third order which I gave you can be written in eight different ways, but they are not really very different, since the same sets of three occur in each one. Here are two of these different forms, and you can find the others.

8	3	4
1	5	9
6	7	2

4	3	8
9	5	1
2	7	6

But there are 880 entirely different squares of the fourth order, which lead to 7040 different forms by writing the sets of numbers in different ways, turning rows into columns, for instance, as we have just done with the one of the third order. The total number of magic squares of the fifth order is more than 750,000.

There is a magic square of the fourth order which has such unusual properties that I shall show it to you. It belongs to a certain class called pandiagonal, *pan* being the Greek word for *all*. You have heard of Pan-American conferences, which are planned for all American countries. A pandiagonal magic square must give the same sum, not only on the two main diagonals, but also on what are known as the broken diagonals. Here is a picture of all the diagonals of a square of the fourth order that run upward to the right.



The four *d*'s form the usual diagonal reading upward from left to right, the four *a*'s form a broken diagonal and so do the *b*'s and the *c*'s.

You see that there are four numbers in each diagonal, broken or whole, and there are just as many diagonals drawn upward to the left. It seems, then, a large order to have all these diagonals satisfy the usual conditions, instead of only two of them.

Sometimes these pandiagonal squares are called perfect, sometimes diabolic. The Hindus first investigated magic squares more than two thousand years ago, and these special squares are often called Nasik, from a town in India where they were investigated by an Englishman who lived there. This special

Nasik square which I shall now show you is carved in Sanskrit characters on the gate of the fort at Gwalior, India.

15	10	3	6
4	5	16	9
14	11	2	7
1	8	13	12

You will find that the sum 34 comes not only from the rows and columns but also from all the diagonals, whole or broken, and, still more surprisingly, from every small square that can be made from this square by taking adjacent numbers, for instance,

15	10		10	3		4	5		
4	5	,	5	16	,	14	11	,	etc.

See if you can find twenty-five different ways in which this unusual square gives a sum of 34, instead of the ordinary ten ways.

A magic square need not necessarily be made up of consecutive numbers from 1 on. Any square arrangement of numbers in which the sum of the numbers in each row, column, and diagonal is the same is called magic. Here, for instance, is a magic square. Do you see how it is made from the regular magic square of the third order?

10	3	8
5	7	9
6	11	4

In Albrecht Dürer's famous picture called "Melancholia" there is a magic square of the fourth order hanging on the wall. An interesting thing about it is that the two middle numbers of the lowest row are 15 and 14, and the date of the picture is 1514.

In ancient times these squares were thought to have magical properties, and they were engraved on metal or stone, and were worn as amulets to ward off evil spirits. They are still used in India for this purpose.

An old puzzle which might make one think of magic squares is the Fifteen Puzzle, which would be very useful in experimenting with squares of the fourth order. It is often for sale in toy or puzzle shops. A shallow, square-bottomed box, just large enough for sixteen blocks, contains only fifteen, labeled 1, 2, 3 · · · 15. They are placed in the box in haphazard order, and the puzzle is solved when, by shifting the blocks about, without taking any out of the box, they are arranged in regular numerical order, 1, 2, 3, 4, etc. It is really quite a puzzle, and it cannot always be done if the blank space is left at the end. Sometimes, in spite of all our efforts, it comes out with all the numbers in order except that it ends in 15, 14, instead of 14, 15. In this case it still can be done, that is, the blocks can be arranged in regular numerical order, if the vacancy occurs in the first space instead of the sixteenth. This fact is not generally known by those who have worried over this puzzle.

A certain mathematical form which resembles these squares is known as a determinant. Magic squares are often written framed by lines on all sides, or even with sets of parallel lines separating the numbers from each other. That is a matter of individual choice. But a determinant is always written with a vertical bar at each side, thus, —

$$\begin{vmatrix} 3 & 5 & 7 \\ 6 & 1 & 9 \\ 8 & 3 & 5 \end{vmatrix}$$

The rows and columns do not have to give the same sum, and in general they look less interesting at first sight than magic squares. But they are remarkably useful in advanced as well as in rather elementary mathematics. However, that is another story.

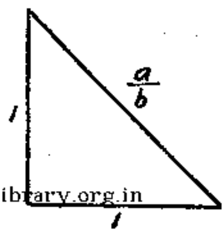
CHAPTER VIII

A FEW REMARKS ON MEASURING AND ON INCOMMENSURABLE NUMBERS

What does it mean to measure a straight line? It means that we take some convenient length which will not change while it is in use, and mark it off on the line as many times as the length of the line allows. Perhaps we choose an inch length to measure with, and find that when we reach the end of the line which we are measuring we have marked off the inch 15 times. Then the line is 15 inches long. But perhaps there is still a piece left over, less than an inch in length. Then we might divide the inch into two equal parts, and see whether one of these parts will complete the measurement. If something still remains, we might take quarter-inch measurements, or eighth-inch, etc. We might finally find the measure of the line to be $15\frac{7}{8}$ inches.

But is it possible that the measuring will never come to an end? That is, we measure in inches and something is left over, in half inches, quarter inches, etc. May it happen that, no matter how long we keep this up, something always remains unmeasured? You can see that in actual practice this can hardly happen if we are working with the unaided eye. For, by the time that we get beyond thirty-seconds or sixty-fourths of an inch the differences are almost imperceptible. But is it really possible in every case to measure a line exactly in terms of any given unit — an inch, for instance, or fractions of an inch? The answer, as you have probably proved in Geometry, is an emphatic *No.*

Suppose you draw a right triangle with its perpendicular sides each one inch long. How long is the hypotenuse? If it can be expressed exactly in inches or fractions of an inch, suppose that it measures $\frac{a}{b}$ inches, both a and b being integers, and the fraction reduced to its lowest terms. Do you see that this is possible if the measure is exact?



Now our best known theorem tells us that

$$\frac{a^2}{b^2} = 2, \text{ or } a^2 = 2\,b^2.$$

It follows that a^2 is an even number, and so a must also be even. If a is even, it must be 2 times something, so let $a = 2\,c$, and substitute $2\,c$ for a:

$$4\,c^2 = 2\,b^2, \text{ or } 2\,c^2 = b^2.$$

This means that b^2 is even, and so, of course, b is even. Since a and b are both even, the fraction $\frac{a}{b}$ is not in its lowest terms, but this contradicts the hypothesis, and therefore the hypotenuse cannot be exactly measured by a fraction. This proof is due to Euclid.

This proof means that, no matter how small a unit you take — a millionth or a billionth of an inch — you can never measure that hypotenuse exactly. This fact we express by saying that the side and diagonal of a square are incommensurable, that is, have no common measure. Any length that exactly measures one of them cannot exactly measure the other. The hypotenuse of this triangle is represented by the symbol $\sqrt{2}$, and you know that, no matter how long a time we may spend in extracting that square root, $1.4142 \cdots$, we shall never come to a point where we can drop our pencil and say, "There! That is the exact answer."

But if we can never express the square root of 2 exactly, why do we say that $\sqrt{2} = 1.414 \cdots$? This number represents an attempt to get as near as possible to a number which, when squared, gives 2 as a result. Since $1^2 = 1$ and $2^2 = 4$, we know that this number lies between 1 and 2. If now we square $1.1, 1.2, 1.3, \cdots$, we shall find that one of these squares is less than 2, while the next is greater than 2. We thus find that $1.4^2 = 1.96$, which is less than 2, while $1.5^2 = 2.25$, which is greater than 2. Next we try 1.41, 1.42, and find that the number we are seeking lies between them. You see that we are really carrying out the process which is called extracting the square root of 2 just by experimenting, not having any definite rule for doing the work. The farther out we go, the nearer the square of the number comes to 2. It is interesting to watch the approach when the numbers are written out in full. Those smaller than 2 are on the left, those larger than 2 on the right.

1^2	$= 1$	2	2^2	$= 4$
1.4^2	$= 1.96$	2	1.5^2	$= 2.25$
1.41^2	$= 1.9881$	2	1.42^2	$= 2.0164$
1.414^2	$= 1.999396$	2	1.415^2	$= 2.002225$
1.4142^2	$= 1.99996164$	2	1.4143^2	$= 2.00024449$

While it is impossible to arrive thus at a number which when squared gives exactly 2, we may come very close to it by continuing this process, getting at each step two numbers between the squares of which 2 lies, and which keep coming closer and closer to each other at each step. These successive numbers are called approximations to the square root of 2. Not only do these numbers never come to an end, but the digits never repeat in regular order, for reasons found in Chapter V.

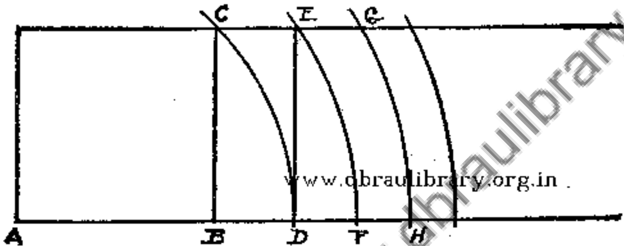
Cube roots may also be found by this method, but of course it takes much longer.

Besides square roots, cube roots, etc., you have probably made the acquaintance of logarithms, and of sines, cosines, and tangents of angles. Most of these numbers are incommensurable, that is, they cannot be expressed exactly as a fraction or a mixed number, which means that they are not repeating decimals.

The same is true of that famous number π, perhaps the most famous number in the world, though a number called e is a rival, and may be considered by some even more famous and useful. You will hear more about π in Chapter IX.

Do you know that it is possible to represent geometrically the square root of any integer? We have seen that, if the side of a square is 1, its diagonal is $\sqrt{2}$. Similarly, if one side of a right triangle is 1 and its hypotenuse is 2, the third side is $\sqrt{3}$, etc.

Here is an interesting way to make a scale which gives the square root of as many integers as we please.



AB is the side of a square, and represents 1 unit. With A as center and the diagonal $AC = \sqrt{2}$ as radius strike an arc cutting AB produced in D. At D erect a perpendicular cutting the opposite side of the square produced in E. With A as center and AE as radius strike an arc cutting AB produced in F, and continue this process. Then $AB = 1$, $AD = \sqrt{2}$, $AF = \sqrt{3}$, $AH = \sqrt{4}$, etc.

You will find that using graph paper you can measure, without drawing any lines at all, $\sqrt{2}$, $\sqrt{5}$, $\sqrt{10}$. What others can you add to this list?

Further, you will find that additional lines are needed to measure on graph paper $\sqrt{3}$, $\sqrt{6}$, $\sqrt{7}$. What others can you add to this list?

CHAPTER IX

SOME FACTS ABOUT π

The number which we call π attracted the notice of mathematicians thousands of years ago. When they tried to measure the area of a circle or the length of its circumference, they found many difficulties in the way. For the fact that the boundary is curved, the very thing that makes it a circle, makes it hard to measure. It was discovered quite early that there is a connection between the circumference of a circle and its diameter, that if the diameter is increased to two or three times its original length, the circumference also becomes just so many times as long. And so the important question arose: Just what is this relation between the circumference and the diameter of a circle? or in other words: What is the constant number that is given by the relation $\frac{\text{circumference}}{\text{diameter}}$?

You could get some idea of its value by rolling a circle on a sheet of paper along a straight line. Mark a point on the rim of the circle, and measure carefully the distance on the straight line between the two points at which this mark on the circle touches the paper. If the number of inches in this distance is divided by the number of inches in the diameter, the result gives an approximate value of this ratio, and if you try the same experiment with a circle of different size, the result should be about the same. The cover of a baking powder tin or a round bottle can be used for the purpose. Or, instead of rolling the object, you can measure the length of a narrow strip of paper that just goes around it.

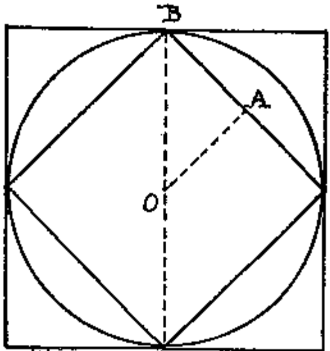
Of course these measurements are very rough, but they may be more accurate than some of the early attempts to find this ratio. For instance, we are told that a great basin in Solomon's temple measured 10 cubits across and 30 cubits around (I Kings, 7 : 23).

The Egyptians, earlier than 1700 B.C., found the value $(\frac{16}{9})^2 = 3.160 \cdots$.

Ptolemy, the famous astronomer, in the second century A.D. gave as the value of this ratio $3\frac{17}{120}$.

And so there came to be more and more accurate estimates as the centuries went by. Archimedes' efforts to compute this value are especially interesting, and you may like to see how it can be done by a method similar to his. But when we remember that Archimedes had no such means of computing with figures as we have, it is quite wonderful that he was able to prove that this ratio lies between $3\frac{10}{70}$ and $3\frac{10}{71}$, that is, between 3.1428 and 3.1408.

Take a circle with diameter equal to 1 unit, and draw two squares, one inscribed and one circumscribed. Represent the circumference of the circle by C, the perimeter of the smaller polygon by p, and that of the larger by P. Then it is clear that $p < C < P$. Now let us measure p and P, as we can easily do, since

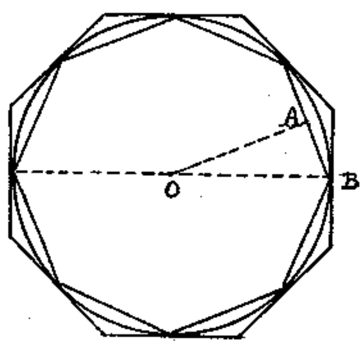


$$AO^2 + AB^2 = OB^2 = \tfrac{1}{4},$$
$$AO = AB,$$
$$AB^2 = \tfrac{1}{8},$$
$$AB = \tfrac{1}{4}\sqrt{2}.$$

This means that $p = 8\,AB = 2\sqrt{2} = 2.828\cdots$. We have then this result from the two squares,

$$2.828 < C < 4.$$

Now bisect the arcs which subtend the sides of the inner square, and construct two regular octagons, one inscribed and one circumscribed. C evidently lies between the two perimeters. We can measure these perimeters quite readily by Trigonometry. In the inner octagon, for instance,



$$\text{angle } BOA = \frac{360^\circ}{16} = 22^\circ\ 30'.$$

Knowing an acute angle and the side OB, we can solve for AB, and p in this case is $16\,AB$. This work takes time, but it is not difficult to get successive values of

p and P as the number of sides is doubled. Here is a table that gives their values to 5 decimal places, using polygons of 4, 8, 16, · · · 512 sides.

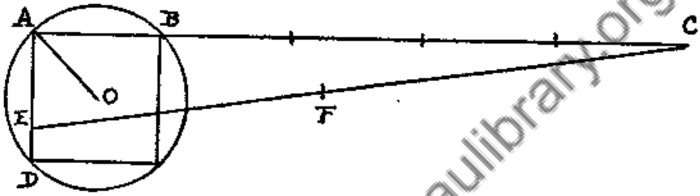
Number of Sides	p	P
4	4.00000	2.82843
8	3.31371	3.06147
16	3.18260	3.12145
32	3.15172	3.13655
64	3.14412	3.14033
128	3.14222	3.14128
256	3.14175	3.14151
512	3.14163	3.14157

Now if you recall that C is always between p and P, you will see that we have an approximation that is correct to two decimal places when we have carried out the bisecting process only four times. Of course it is more accurate the longer we continue this process. It was only the fact that Archimedes had a very clumsy notation that kept him from getting a surprisingly close value of this ratio. Since we took the diameter as 1, we have found that this ratio is approximately 3.1416.

As is often the case with a new arrival in the family, there was some difference of opinion as to what this ratio should be called. Some called it c because of its connection with the circle or the circumference; some called it p, the initial of perimeter; but finally we seem to have agreed to call it π, the Greek equivalent of the Latin p, and the first letter of the Greek word *perimeter*. The letter π was first used for this purpose in the 17th century.

There are many methods by which an approximate value of π may be constructed with rather simple geometric figures. Some of these may interest you.

Let the radius of the circle with center at O be 1. Inscribe in it a square. The side $AB = \sqrt{2}$. On AB



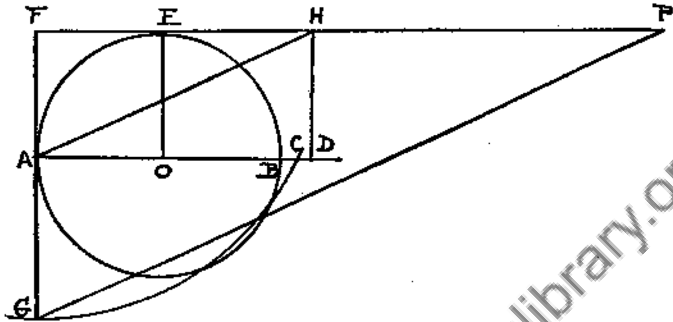
produced lay off AB four times. Then $AC = 5\sqrt{2}$. On AD lay off $AE = AO = 1$, and join E and C. From C on CE lay off $CF = 4$.

$$EC^2 = EA^2 + AC^2 = 1 + 50,$$
$$EC = \sqrt{51} = 7.141428 \cdots.$$
Therefore, $$EF = \sqrt{51} - 4 = 3.141428 \cdots.$$

This value of EF differs from the true value of π by less than .000165.

Here is another rather easy construction which gives a still more accurate value of π.

Draw a circle of diameter 1. Let AB be any diameter and OE a radius perpendicular to it. Tangents drawn at A and E intersect at F. On AB produced lay off $BC = \frac{1}{10}$ and $BD = \frac{2}{10}$. At D draw DH perpendicular to FE and meeting it at H. Join A and H. With F as center and radius FC cut off FG on FA produced, and through G draw a line parallel to AH, meeting FH in P.



Then

$$\frac{GP}{AH} = \frac{FG}{FA}.$$

But

$$AH^2 = AF^2 + FH^2 = (\tfrac{5}{10})^2 + (\tfrac{12}{10})^2 = \tfrac{169}{100}.$$

Therefore

$$AH = \tfrac{13}{10},$$
$$FG^2 = FC^2 = FA^2 + AC^2 = (\tfrac{5}{10})^2 + (\tfrac{11}{10})^2 = \tfrac{146}{100}.$$
$$FG = \tfrac{1}{10}\sqrt{146},$$
$$FA = \tfrac{1}{2}.$$

Therefore, on substituting these three values in the first equation and solving for GP,

$$GP = \tfrac{13}{50}\sqrt{146} = 3.14159\cdots.$$

This is a remarkably close approximation.

Just as with ruler and compass we can mark off lengths corresponding to $\sqrt{2}$, $\sqrt{3}$, etc., as accurately as mechanical constructions allow, so, by the use of more elaborate machinery, the distance π may be measured off. One such machine is called an integrator.

Different values of π are used according to the accuracy needed in the result. For elementary use $\pi = \frac{22}{7}$ does very well. One rarely needs a more accurate value than 3.1416. But the value of π has been computed to hundreds of decimal places, far more than could possibly be needed in any computation. One man, Ludolph van Ceulen, computed π to 35 decimal places, and the result was carved on his tombstone in Leyden. In German books one often sees π referred to as "Ludolph's number." An Englishman named William Shanks found the value to 707 decimal places.

One reason why so much time and energy, worthy of a better cause, were spent on these long computations was that men thought that if they only went out far enough they would find that the digits would at last begin to repeat, and if they could only find a repeating decimal, they would have an exact expression for the value of π in fractional form. But comparatively recently it was proved that π is an incommensurable number, so that even if any of you were ambitious to beat Shanks' record, and should carry π out to 1000 decimal places, the digits would follow each other without ever beginning a regular repetition.

Some people who are interested in such reckoning do not know this fact, and are still trying to find an exact value for π. Such people are generally called "circle squarers," because if one could find π exactly, one would have an exact expression not only for the circumference of a circle, but also for its area. In other words, one could solve the old problem — To draw a square equal in area to a given circle.

Several rather clever schemes for remembering the value of π have been devised. The simplest is "Yes, I have a number," in which the number of letters in the successive words — 3, 1, 4, 1, 6 — gives the digits of π to 4 decimal places. A similar scheme gives π to 12 decimal places:

"See, I have a rhyme assisting
My feeble brain, its tasks ofttimes resisting."

A still longer French verse gives π to 30 decimal places:

"Que j'aime à faire apprendre un nombre utile aux sages!
Immortel Archimède, sublime ingénieur,
Qui de ton jugement peut sonder la valeur?
Pour moi ton problème eut de pareils avantages."

Have you ever heard that π is useful for anything besides measuring the circle? The fact that it enters into this measurement makes it seem quite natural that we should find it in other rounded areas, like the ellipse, and in volumes like the sphere, cone, cylinder, and many other forms. But in reality π has far more important uses than its connection with the circle, and it occurs in many other subjects besides Geometry. In Calculus, and other branches of Mathematics which belong to that great division known as Analysis, π is constantly putting in an appearance. For instance, an important but very simple-looking series,

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$

proves to come as near as we please to the value $\frac{\pi}{4}$ by combining enough of its terms. Also the series

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots$$

can be made to come as close as we please to $\frac{\pi^2}{6}$ by combining a sufficient number of its terms.

One important use of π occurs in what is known as Probability. If you should look up this subject in the *Encyclopedia Britannica*, you would be surprised to see how often π occurs in the article.

A famous English mathematician, De Morgan, was once explaining to a friend a mathematical formula which gave the probability that the number of persons in a large group who would be alive after a certain number of years would lie between two given numbers. The symbol π occurred in the formula, and De Morgan, when asked what it meant, gave the usual definition — the ratio of the circumference of a circle to the diameter. His friend was much surprised, and was sure that De Morgan must be mistaken in thinking that there could be any connection between π and the number of persons alive at a given time.

There is a way to determine π roughly which depends upon its occurrence in probability formulas. Take a series of parallel lines, each k units from the next, drawn on a large sheet of paper, or perhaps the boards in the floor are regular enough in width to serve the purpose. Take a stick of length t, shorter than k, the distance between the lines, and drop it on these lines again and again, keeping a careful record of the number of times you try the experiment and the number of times the stick falls so as to lie across one of the lines. The formula for the probability, p, that it will cross a line is

$$p = \frac{\text{number of times it crosses a line}}{\text{number of experiments}} = \frac{2\,t}{k\pi}.$$

If you find a value for p by carrying out this experiment a good many times, you can compute π from this formula, $\pi = \frac{2\,t}{kp}$. Several men have tried this experiment. One made 1120 trials and found π equal to 3.1419. As a general thing the greater the number of trials the more nearly correct the result will be.

Here is another simple method of computing π from a probability formula. If two numbers are written down at random, the probability that they will be prime to each other is $\frac{6}{\pi^2}$. This means that about 6 out of 9 of such pairs are likely to be prime to each other, or, more accurately, 600 out of 987. Fifty students once tried this, each writing down rapidly, without stopping to think of their factors, 5 pairs of numbers. It was found that 154 pairs were prime to each other. That is, $\frac{6}{\pi^2} = \frac{154}{250}$ is approximately true. This gives $\pi = 3.12$. This experiment could probably be tried more easily than the preceding one.

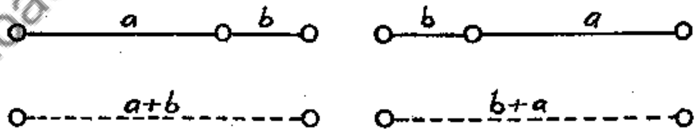
CHAPTER X

GEOMETRICAL ARITHMETIC

In your work in Plane Geometry you have learned to solve many problems which sound as if they belong in an Arithmetic, except that you are working with segments of lines instead of numbers. I shall make a collection here of some of these geometric solutions of arithmetical problems, most of which you have met already, but some may be new to you.

Addition. To find the sum of two given segments, a and b.

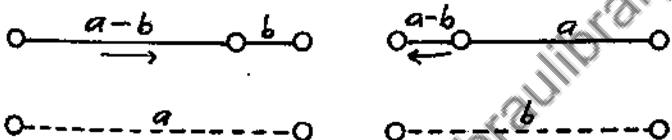
This means that a single segment is to be found equal in length to the combined lengths of a and b, a very easy thing to do.



The process is sometimes described by saying that the initial point of b is placed on the end point of a, both lines going in the same direction. The sum of the two extends from the initial point of a to the end point of b.

In the first figure b is added to a in this way, in the second a is added to b. The resulting length is evidently the same in each case. If the lengths of a and b are measured, and the results added, the sum should be the same as the measure of the length of $a + b$, the same unit of measure being used in each case.

Subtraction. To find the difference between two given segments, a and b.



This means that a third segment is to be found, such that, if b is added to it, the sum is equal to a.

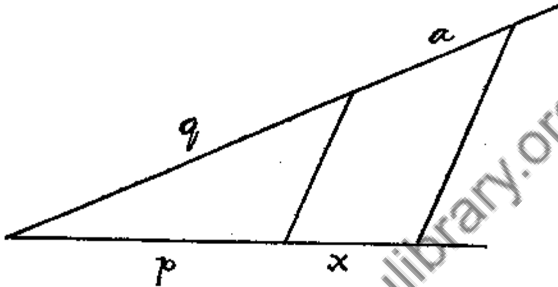
Let a be drawn in the positive direction, and from its end point let b be drawn in the opposite direction. The result is again the distance from the initial point of a to the end point of b. In the first figure a is longer than b, in the second b is longer than a, and the result, being negative, is read in the direction opposite to that of a.

Multiplication.

1. To multiply a segment a by a given integer n. This amounts to laying off the segment a along a straight line n times.

2. To multiply a segment a by a given fraction $\frac{p}{q}$.

Let $x = \frac{ap}{q}$, then $\frac{x}{a} = \frac{p}{q}$, and the construction can be made as follows, using similar triangles.

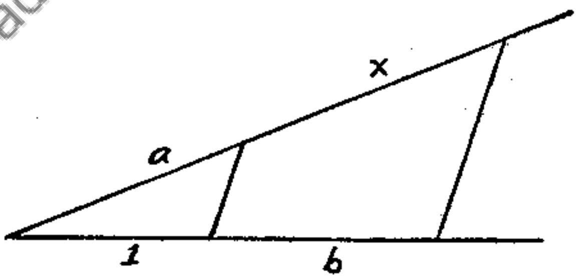


If $p = 1$, we have solved the problem — To divide a segment by an integer.

3. To multiply a segment a by another segment b.

This really means finding a segment whose measure in terms of some unit shall be the same as the product of the measures of the two given segments, the same unit being used throughout.

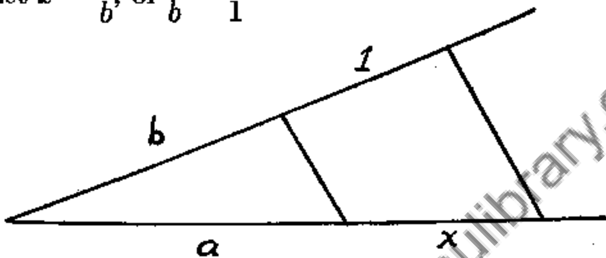
Let $x = ab$, then $\frac{a}{1} = \frac{x}{b}$, and the solution is readily carried out by similar triangles.



Division. To divide one segment a by another segment b.

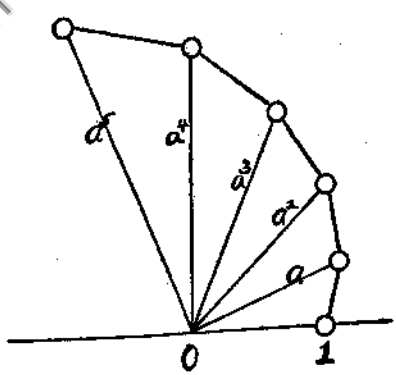
This means, finding a segment whose measure is the quotient of the measures of the two given segments.

Let $x = \frac{a}{b}$, or $\frac{a}{b} = \frac{x}{1}$.

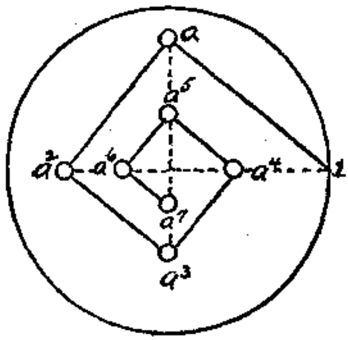


Involution. Given a segment a, to find segments which shall represent a^2, a^3, etc.

This is merely an extension of multiplication. Since the required proportion now has the form $\frac{1}{a} = \frac{a}{x}$, there is a good way to find the successive segments which represent powers of a. Using a protractor, you can lay off quite accurately several equal angles of any convenient size with the same vertex, the terminal line of each being the initial line of the next. On the first initial line mark off one unit, on the next line



a units, and complete the triangle. The second triangle, with one side a, is to be made similar to this one. The figure makes the construction and the results clear.



The work is simpler if right angles are used, for then each new line is drawn perpendicular to the preceding one. Why?

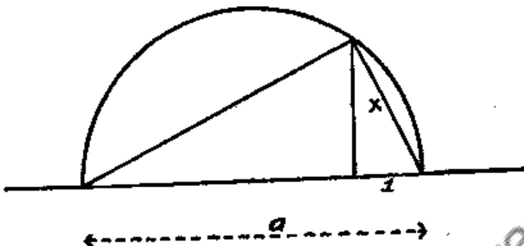
If a is greater than 1, the lines a^2, a^3, $\cdots$ keep growing longer, while if a is less than 1, they grow shorter. Do the figures show clearly that, if $a = 1$, all the powers of a are the same?

Evolution. To extract the square root of a given segment a.

This amounts to solving the proportion $\frac{a}{x} = \frac{x}{1}$, that is, x is to be a mean proportional between a and 1. We recall two theorems about a mean proportional, each connected with a right triangle, and we shall try them both.

1. If a perpendicular is dropped on the hypotenuse of a right triangle from the opposite vertex, either side of the triangle is a mean proportional between the whole hypotenuse and the adjacent segment.

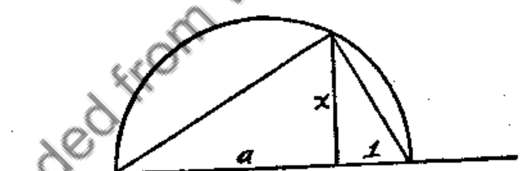
We must then use a and 1 as the hypotenuse and the segment laid off on it as in subtraction. Whichever is the larger will, of course, play the part of the hypotenuse.



The construction is readily carried out. Since $\frac{1}{x} = \frac{x}{a}$ $x = \sqrt{a}$.

2. The perpendicular let fall on the hypotenuse of a right triangle from the opposite vertex is a mean proportional between the segments of the hypotenuse.

This theorem is perhaps easier to use than the other.



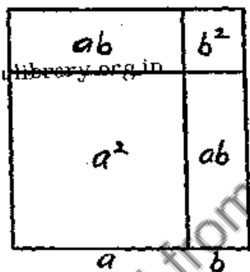
The construction is readily seen from the figure. As before, $x = \sqrt{a}$.

If ruler and compasses are the only instruments to be used, it is impossible to give an exact construction for a cube root or for any higher roots, except, of course, fourth roots, eighth roots, etc. How can you find these?

On page 81 I showed you how you can make a ruler to measure square roots of integers. These roots may also be found by a quite different construction.

Beginning with a right triangle with two sides each equal to 1, its third side is $\sqrt{2}$. Take this as one side of a right triangle whose second side is 1, the hypotenuse is $\sqrt{3}$. This construction may be continued as far as one wishes. How many square roots do you suppose you will have constructed before the drawing begins to overlap the first triangle?

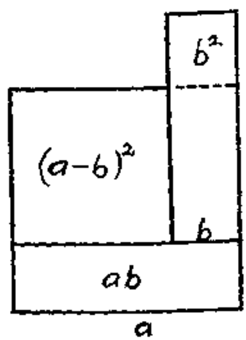
Some algebraic constructions. These next constructions belong to Algebra rather than to Arithmetic.



1. To prove geometrically that $(a + b)^2 = a^2 + 2\,ab + b^2$.

The figure shows very plainly that the square whose side is $a + b$ is made up of the two squares whose sides are a and b and two rectangles with a and b as adjacent sides.

2. The corresponding figure for $(a - b)^2 = a^2 - 2\,ab + b^2$ is perhaps not quite so easy to understand.

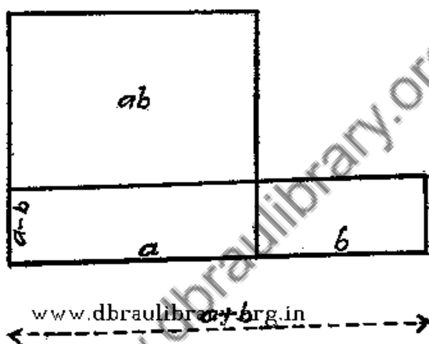


The square on $a - b$ may be obtained by taking first the square on a and adding to it the square on b, then subtracting two rectangles whose adjacent sides are a and b. There are other ways to carry out this construction with which you may like to experiment.

3. To prove geometrically the formula

$$(a + b)(a - b) = a^2 - b^2.$$

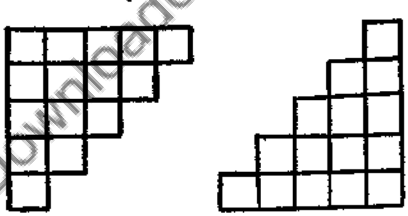
The wide rectangle has its adjacent sides equal to $a + b$ and $a - b$. It may be obtained by taking first the square on a, subtracting from it the rectangle whose adjacent sides are a and b, and adding to this the rectangle whose adjacent sides are $a - b$ and b. That is, $(a+b)(a-b)$ $= a^2 - ab + (a - b)b$ $= a^2 - b^2$.



To find geometrically the sums of certain combinations of integers.

1. To find the sum of the arithmetical progression $1 + 2 + 3 + 4 + \cdots + n$.

Let each unit be represented by a square. Then the sum of any number of terms may be arranged in a sort of stair. Here is a drawing of the sum of 5 terms, arranged in two different orders. These two figures can be fitted together so as to form a rectangle which measures 5 by 6 units, that is, it contains 30 square units. Since this gives the sum of the first

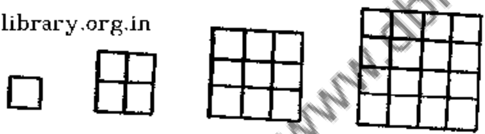


5 terms twice over, the sum must be 15. That is, $1 + 2 + 3 + 4 + 5 = 15$.

If another term, 6, is added, the rectangle measures 6 by 7, and the sum of 6 terms is seen to be $\frac{6 \times 7}{2} = 21$. And, in general, we see that if n terms are added, the sum will be $\frac{n(n+1)}{2}$.

2. To find the sum of the arithmetical progression $1 + 3 + 5 + \cdots (2n - 1)$.

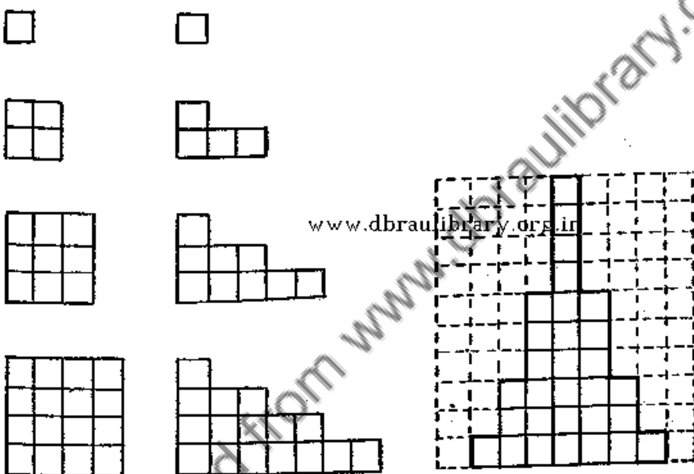
As before, we represent each unit by a square. You will readily see that each sum can be arranged in a square. Beginning with 1, the first figure, we form the second figure by taking 1 and putting 1 on at each of two adjacent sides and at a corner, as shown in the figure, making a square with 2 units on a side. Then 2 units are put on each of two adjacent sides and 1 in the corner, thus adding 5, and making a square of 3 on each side. The next number, 7, will evidently fit in in the same way, on this square; that is, when we have 3^2, we add $2 \times 3 + 1$ to get 4^2, and, in general, when we have n^2, we add $2n + 1$, or the next odd number, and get $(n+1)^2$.



3. To find the sum of the squares of the first n integers, that is,

$$1^2 + 2^2 + 3^2 + 4^2 + \cdots + n^2.$$

Let us arrange the small squares which stand each for one unit first in squares, and then as the sum of consecutive odd numbers, as shown in the first two columns. Here we see that the single square occurs as many times as the number of terms we are using, in this case 4 times. A row 3 units long occurs once less, in this case 3 times. A row 5 units long occurs twice,



and one 7 units long occurs once. So we rearrange the whole thing once more in a sort of skyscraper effect. Now let us fit in the unit squares about this so as to form a rectangle, as shown in the last figure.

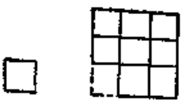
On each side the same number of squares has been added, and you will see that, adding from the bottom, we have fitted in on each side 1 square, then 4 squares, then 9 squares, then 16 squares; that is, in this last figure the sum which we set out to find, $1^2 + 2^2 + 3^2 + 4^2$, occurs 3 times. But this rectangle measures 9×10, and so the sum of these four squares is 30, as is easily verified.

To go into greater detail, the height of the rectangle is $1 + 2 + 3 + 4$, and its width $2 \times 4 + 1$. If the next square is added on in the same way, the height will now be $1 + 2 + 3 + 4 + 5$, and the width $2 \times 5 + 1$, as you can easily show by sketching the figure. And so we see in general that, if the squares of the first n numbers are added, the area of the rectangle is measured by the base, $2n + 1$, times the altitude, $1 + 2 + 3 + \cdots + n$. This last form we have shown to be equal to $\frac{n(n+1)}{2}$. And so a general formula for the sum of the squares of the first n integers is $\frac{n(n+1)(2n+1)}{6}$. This formula was suggested for verification in Chapter VI.

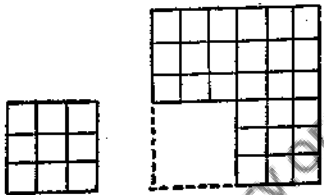
4. To find the sum of the cubes of the first n integers, that is,

$$1^3 + 2^3 + 3^3 + \cdots + n^3.$$

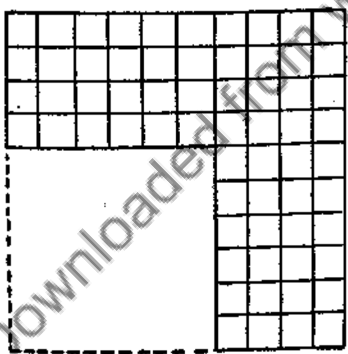
This is much easier than we might naturally expect it to be. Using the same representation, 1^3 is represented by a single square, 2^3 by 8 squares, which may be arranged as in the second figure. These can be fitted together so as to form a square, except for a hole in one corner, and so it is clear that $1^3 + 2^3 = 3^2$.



You will see at once that 3^3 or $27 = 3 \times 9$, and these three squares can be arranged in such a position that the former square just fits in; that is, $1^3 + 2^3 + 3^3 = 6^2$.



Now we will arrange $4^3 = 64$. As before, take out $4^2 = 16$, leaving 48, and 48 unit squares can be arranged in two rectangles, each measuring 4×6. And so the figure leaves just room enough for the square which we had before to slip in, and we see that $1^3 + 2^3 + 3^3 + 4^3 = 10^2$. This makes us suspect that this large square, 10 units on a side, will fit into the space left when we arrange $5^3 = 125$ units in the same general way. Let us experiment with it. Take out $5^2 = 25$ for the corner square. This leaves 100 to be arranged in two rectangles on the two sides of the square, which is readily done if each rectangle measures 5 by 10, and so the square above fits in exactly, completing a still larger square with 15 units on a side.



The sides of these successive squares which we have built up are 1, 1 + 2, 1 + 2 + 3, 1 + 2 + 3 + 4, 1 + 2 + 3 + 4 + 5. And it is true that, if we continue to any desired number of terms, n, the sum will be

$$1^3 + 2^3 + 3^3 + \cdots + n^3 = \left[\frac{n(n+1)}{2}\right]^2.$$

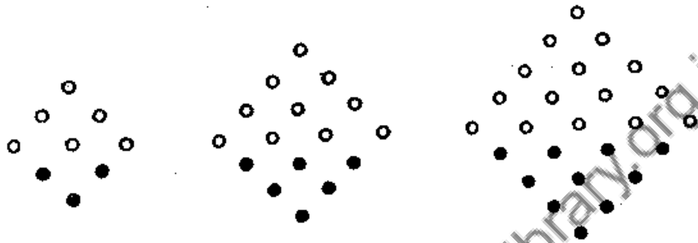
Triangular numbers.

The numbers 1, 2, 3, 4, $\cdots$ are sometimes arranged in what is called a triangular formation, as shown below, and 1, 3, 6, 10, $\cdots$ are therefore called triangular numbers, since forms that represent them can be arranged in triangles. We have already a formula for triangular numbers, since we know that $1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$. For each value that we may give to n, $n = 1, 2, 3$, etc., we get from this formula a triangular number. The properties of these numbers were studied

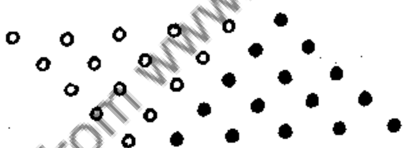
by Pythagoras and his followers. You may have noticed that flocks of birds often fly in this triangular or wedge-shaped formation, and scientists have found that such an arrangement makes a long flight much easier. Perhaps you have seen several airplanes when flying together arranged in this same formation.

The sum of two consecutive triangular numbers is always a square, as you can easily prove by Algebra. It may also be shown to be true by Geometry, by fitting the two triangular forms together, as the figures show.

The general formula for triangular numbers may also be found geometrically. Arrange the circles or dots which represent any one of these numbers in the triangular form, placing beside it the same triangle inverted, thus forming a parallelogram. The one drawn here contains 5 rows of 6 units each; that is, 30 units.



The number corresponding to either triangle is therefore $\frac{5 \times 6}{2} = 15$. This construction is quite general. If any such triangle with base n has a congruent triangle

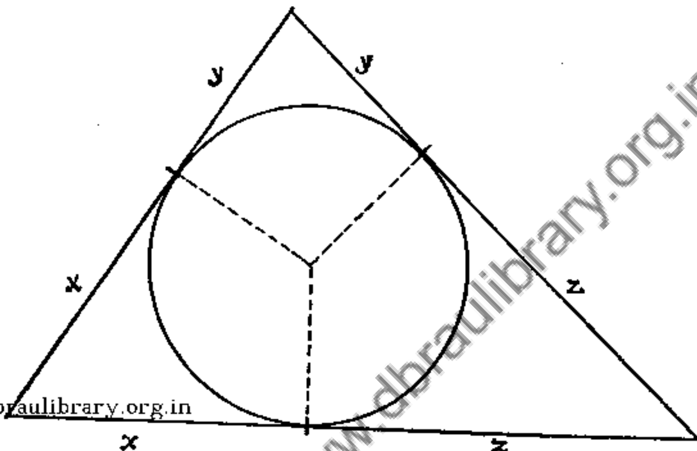


inverted and placed beside it, the parallelogram which is thus formed contains n rows of $n + 1$ units each, so that the number represented by the given triangle is $\frac{n(n+1)}{2}$.

The following geometric construction solves a problem which might be worked algebraically with three unknown quantities and therefore three equations.

Find three numbers such that the sum of the first and second is 50, the sum of the second and third is 60, and the sum of the third and first is 70.

Construct a triangle with sides 50, 60, and 70, and in it inscribe a circle. Since tangents to a circle from



any point are equal, the figure may be lettered as shown, x, y, and z each occurring twice. But these values, x, y, z, are evidently the solution of the problem.

PROBLEMS

1. Prove that if a triangular number is multiplied by 9, and 1 is added to the product, the result is a triangular number. In other words, if t is a triangular number, so is $9t + 1$.

This is proved by taking the formula derived for a triangular number, $\frac{1}{2}n(n+1)$, and showing that $\frac{9}{2}n(n+1)+1$ is also triangular. Reduce this form to a fraction and factor the numerator. Notice that all that is necessary to prove a number triangular is to show that when the denominator is 2, the numerator is the product of two consecutive numbers. For example, $\frac{1}{2}(3n+1)(3n+2)$, $\frac{1}{2}(5n-3)(5n-2)$, $\frac{1}{2}(8n+6)(8n+7)$ are triangular numbers.

2. If 9 is chosen as the base, prove that 1, 11, 111, etc., are triangular numbers. Find a connection between this problem and the preceding.

3. If 9 is chosen as the base, prove that when 1 is annexed to any triangular number, the result is a triangular number. For example, 6 is a triangular number, show that 61 is triangular; 16 is a triangular number, show that 161 is also triangular.

4. If 3 is chosen as the base, prove that when 01 is annexed to a triangular number, the result is triangular.

5. If 5 is chosen as the base, prove that when 03 is annexed to a triangular number, the result is triangular.

By this general method triangular numbers may be obtained from any triangular number written to an odd base, $2k+1$. Multiply by $(2k+1)^2$ and add $\frac{1}{2}k(k+1)$. It is not hard to prove this rule correct.

CHAPTER XI

ODDITIES OF NUMBERS

The numbers that we shall deal with in this chapter are not all odd numbers, but they do odd things or have odd characteristics, as you will see. It will, I think, give you more amusement if I do not work out everything in full. It is generally pleasanter to see a process going on than to look at the finished result. That is one reason why most young folks like to use their hands, and enjoy making things themselves.

1. Some facts about 37.

Take the arithmetical progression 3, 6, 9, 12, 15, · · · and multiply each term by 37.

One number which solves the following problem is 37. Find a number of two digits which, when multiplied by the sum of its digits, gives a product equal to the sum of the cubes of its digits. This is not easy to solve, but you can see that 37 meets the conditions. Can you find a second answer? 48.

2. Multiply 230,769 by 341.

```
  230769
     341
--------
  230769
 923076
692307
--------
```

Do you see how unusually easy it is to add up the columns?

3. Multiply 142,857 by 326,451.

Here again you will see something unusual about the columns to be added. The partial products make a sort of parallelogram. Notice the digits running up its right side, 142 · · · . Does that sound familiar?

4. Multiply 76,923 by 68,275.

5. Multiply 143 by 161 or 196 or 812 or, in general, by any multiple of 7 not greater than 6993, and see how the digits follow each other. The reason can be found without much difficulty if you notice the connection between the digits in the product and the number of times 7 goes into the multiplier.

6. How to get practice in writing a certain digit.

"Which digit do you write most illegibly?" you ask a friend. Perhaps he replies "Five." Then tell him to take the number 12345679 and multiply it by 45. In each case the multiplier is to be 9 times the number mentioned.

7. Some numbers may be divided into four parts which have odd properties. For instance, 100 may be divided into four parts such that, if one is increased by 4, one diminished by 4, one multiplied by 4, and one divided by 4, the result in each case is the same.

$$12 + 4 = 16,$$
$$20 - 4 = 16,$$
$$4 \times 4 = 16,$$
$$64 \div 4 = 16.$$

This same kind of thing can be done with 45. Divide it into four parts, such that, if one is increased by 2, one diminished by 2, one multiplied by 2, and one divided by 2, the results are the same.

You will find that 64, 75, 80, and, in fact, any number that is divisible by a square, can be divided up in this way. For example, 80 may be divided into four parts as follows:

$$18 - 3 = 15,$$
$$12 + 3 = 15,$$
$$5 \times 3 = 15,$$
$$45 \div 3 = 15.$$

The nine digits, with or without zero, can be combined in all sorts of clever ways in writing numbers, as the next illustrations show.

8. Use the nine digits once each in writing numbers which add up to 225.

One solution is $1 + 25 + 43 + 69 + 87 = 225$. You can find other arrangements which give this same sum.

If the nine digits are to be combined to make integers that shall add up to a certain number, that number must be a multiple of 9. Why?

If other forms are allowed, such as fractions, roots, or any combinations that employ algebraic symbols, the number of possible arrangements is greatly increased, as the following illustrations show.

9. Using the nine digits once each, and zero also once if desired, write numbers whose sum is 100.

One solution is $74 + 25 + \frac{3}{6} + \frac{9}{18} = 100$.

I have found over twenty solutions. There are doubtless many more, using integers and fractions only. Here is one in which roots are used,

$$5 + 10 + 78 + \sqrt[2]{9} + \sqrt[3]{64} = 100.$$

10. Using the nine digits once each and zero if desired, write numbers whose sum is 1.

One solution is $\frac{35}{70} + \frac{148}{296} = 1$.

11. As in No. 10, write a fraction equal to $\frac{1}{2}$.

One solution is $\frac{9327}{18654}$. You can find others.

12. As in No. 10, write fractions equal to $\frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \frac{1}{7}, \frac{1}{8}, \frac{1}{9}$.

As an illustration, $\frac{6471}{58239} = \frac{1}{9}$. There are at least two other fractions with numerator ending in 1 which are equal to $\frac{1}{9}$.

In the following puzzles the symbols $+, -, \times, \div$, ( ), may be used.

13. Writing 1 five times obtain 100 as the result.

$$111 - 11 = 100.$$

14. Writing 3 five times obtain 100 as the result.

15. Writing 5 five times obtain 100 as the result.

16. Writing 5 four times obtain 100.

17. Writing 5 six times obtain 100.

18. Writing 9 four times obtain 100.

19. Writing 9 six times obtain 100.

20. Writing 3 five times obtain 31.

21. 16 is a square. Insert 15 between 1 and 6; 1156 is a square. Insert 15 again in the middle of this number; the result is a square. Insert 15 again in the middle of this number, etc. As is to be expected, the numbers of which these are squares are formed by a regular method.

This may be stated in a general form as follows. Write 1 n times, then write 5 $n - 1$ times, and end with a 6. The result is a square whatever the value of n. See if you can find a general form for the square root of this number.

22. The same kind of statement as in No. 21 can be made about 49. It is a square, and if 48 is inserted in the middle, the result is a square, etc. Show that this is true.

In both these cases the reason why it comes out this way is clearly seen when one actually finds these squares by multiplication.

23. Take any number, write below it the same number, putting the first digit directly under the fourth digit of the chosen number. The sum is always divisible by 7, 11, and 13.

For example:

$$\begin{array}{r} 39842 \\ 39842 \\ \hline 39881842 \end{array}$$

If the chosen number has less than four digits, fill out with zeros.

24. Multiply each of the following nine numbers by 7, and notice the products.

15,873, 31,746, 47,619, 63,492, 79,365, 95,238,
111,111, 126,984, 142,857.

25. Take the "nine times" table, and note the way the digits change:

09, 18, 27, 36, 45, 54, 63, 72, 81, 90.

Each number has its first digit increased by 1 and its second digit decreased by 1 to form the next number. This same thing is true in other products of 9 besides the numbers from 1 to 10. Take any ten consecutive integers ending with a multiple of 10; as, for instance,

241, 242, 243, 244, 245, 246, 247, 248, 249, 250,

and multiply each by 9. The products are

2169, 2178, 2187, 2196, 2205, 2214, 2223,
2232, 2241, 2250.

The same rule holds as before. The last digit decreases by 1 each time, while the number made up of the three digits in front of it increases by 1. This makes it very easy to write a set of such products after the first one is found. For instance, find the product by 9 of each of the ten numbers 361, 362, · · · 370, the first one being 3249.

26. The following squares give some interesting numbers, as you will see if you multiply them out.

$$9^2 =$$
$$99^2 =$$
$$999^2 =$$
$$9999^2 =$$
$$99999^2 =$$

Can you find from this a rule for squaring a number that is made up entirely of 9's?

27. The powers of 11 will recall to you the coefficients in the expansion of $(a + b)^2$, $(a + b)^3$, etc.

$$11^2 =$$
$$11^3 =$$
$$11^4 =$$

You will find that higher powers of 11 will also follow the binomial theorem if in adding the partial products you keep the entire number without carrying anything to the next column, thus:

$$\begin{array}{cccccccc}
 & 1 & 4 & 6 & 4 & 1 & = & 11^4 \\
 & & & & 1 & 1 & & \\
\hline
 & 1 & 4 & 6 & 4 & 1 & & \\
1 & 4 & 6 & 4 & 1 & & & \\
\hline
1 & 5 & 10 & 10 & 5 & 1 & = & 11^5
\end{array}$$

Of course, if the addition were carried out in the usual way, the last sum would be 161051. But if written to a larger base than 10, it would be $15XX51$.

28. Find the following squares.

$$1^2 =$$
$$11^2 =$$
$$111^2 =$$
$$1111^2 =$$
$$11111^2 =$$
$$111111^2 =$$
$$1111111^2 =$$
$$11111111^2 =$$

If this is carried on further, does it continue to lead to the same kind of result?

29. Find the value of

$$0 \times 9 + 8 =$$
$$9 \times 9 + 7 =$$
$$98 \times 9 + 6 =$$
$$987 \times 9 + 5 =$$
$$9876 \times 9 + 4 =$$
$$98765 \times 9 + 3 =$$
$$987654 \times 9 + 2 =$$
$$9876543 \times 9 + 1 =$$
$$98765432 \times 9 + 0 =$$
$$987654321 \times 9 - 1 =$$
$$9876543210 \times 9 - 2 =$$

30. Find the value of

$$1 \times 8 + 1 =$$
$$12 \times 8 + 2 =$$
$$123 \times 8 + 3 =$$
$$1234 \times 8 + 4 =$$
$$12345 \times 8 + 5 =$$
$$123456 \times 8 + 6 =$$
$$1234567 \times 8 + 7 =$$
$$12345678 \times 8 + 8 =$$
$$123456789 \times 8 + 9 =$$

31. Find these products :

$$9 \times 6$$
$$99 \times 66$$
$$999 \times 666$$
$$9999 \times 6666$$

. . . .

Will it work further?

Can you write a formula for the product of two numbers made up of n nines and n sixes respectively?

If in the second number 5 or 7 or any other digit were substituted for each 6, would the results be similar to these?

32. Here are some questions about squares.

Of the squares that contain two digits only, how many are made up of even digits? how many of odd digits?

How about the squares which contain three digits? Do any of them consist of even digits only? or of odd digits only? Zero is to be included among the even digits.

Now let us find all numbers whose squares consist of four even digits. Perhaps you will see at once that 80 must be one of them.

This puzzle will be mostly experiment for you, but some hints will shorten the work.

A number that has four digits in its square must lie between 31 and 100. The square of every number from 32 to 44 begins with 1; the squares of numbers from 55 to 63 begin with 3, from 71 to 77 with 5, from 84 to 89 with 7, from 95 to 99 with 9. These facts reduce greatly the amount of experiment needed. Including 80, there are four such numbers.

33. Find a square whose last two digits are its square root. There are two such squares.

34. Find a square of four digits such that its first two digits and its last two digits taken in order form two numbers which are both squares. There is only one such square, if numbers such as 1600 are not counted.

35. Find three consecutive odd numbers the sum of whose squares consists of four identical digits.

This is quite readily solved. How do you write three consecutive odd numbers? What is the sum of their squares? This sum must be equal to 1111, or 2222, or 3333, etc. Now $12n^2 + 12n + 11$, the result that you will probably get, is an odd number. Why? This rules out four of the possible sums listed above. If 11 is subtracted from any one of the remaining sums, the difference must be divisible by 12. Why? All this makes it easy to find the one number which solves the problem.

Now we turn to some questions about cubes.

36. Take the following sets of numbers, and show that the sum of the numbers in each row is the cube of the number of the row.

(1) 1,
(2) 3, 5,
(3) 7, 9, 11,
(4) 13, 15, 17, 19,
(5)

Hint. Show that the nth row begins with $n^2 - n + 1$, and ends with $n^2 + n - 1$. The rest is then easy, since in each case you are to find the sum of an arithmetic progression.

37. Take the following sets of numbers, and multiply each in the first set by 1, each in the second set by 2, each in the third set by 3, and, in general, each in the nth set by n. Show that the sum of the numbers in each of these new sets is the cube of the number of the set.

(1) 1,	(1) 1,
(2) 1, 3,	(2) 2, 6,
(3) 1, 3, 5,	(3) 3, 9, 15,
(4) 1, 3, 5, 7,	(4) 4, 12, 20, 28,
(5) 1, 3, 5, 7, 9,	(5)

Hint. The nth set on the right begins with $\cdots$, contains numbers, which differ by $\cdots$. Their sum can be found by using the formula $\cdots$

For a different proof use a formula on page 63.

38. Take the following numbers, and show that the sum of the numbers in each group is the cube of the number of the group.

(1) 1,
(2) 3, 5,
(3) 6, 9, 12,
(4) 10, 14, 18, 22,
(5) . . . . .

The first number in each group is a pyramid number, and the numbers in each group are in arithmetical progression. This, of course, as in the questions just before it, is not really proved until we have shown that a general row — the nth — adds up to n^3.

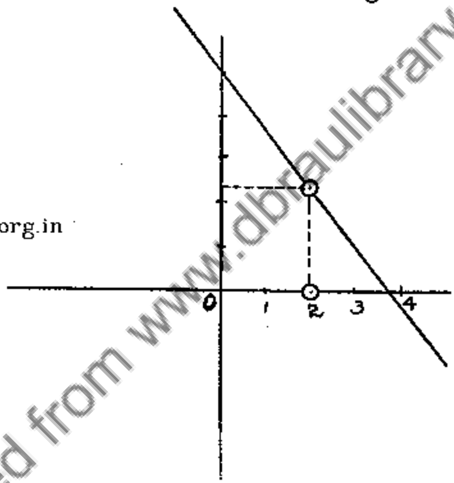
CHAPTER XII

EQUATIONS WITH MANY ANSWERS

You have found that you can always solve an equation of the first degree in x, and that there is just one answer; that if the equation is of the second degree in x, there are always two answers, equal or unequal to each other. You may guess from this that if you knew how to solve an equation of the third degree, you would probably find three answers, etc.

Also you have learned that an equation with both x and y in it needs a second equation to be used with it in order to find a definite pair of values for x and y. There is, however, something that you have learned to do with a single equation in both x and y, and that is to draw its graph. Now this graph is really a picture of all the many pairs of values of x and y which are solutions of the equation. Take, for example, the equation $4x + 3y = 15$. We know from our study of graphs that for every value we give to x we shall find a corresponding value of y, and that, if we plot these points (x, y) in the usual way, they will all lie on a straight line. Also we know that after we have drawn this line carefully we have only to measure off on the x-axis from the origin any distance we may choose for the value of x, and the length of the perpendicular to the x-axis drawn from that point and cut off by the line gives the corresponding y. If, for instance, in the figure we let x be 2, we can see that y has a value between 2 and 3.

Now a question that is often of interest and importance is this: How many points on this line have integral values for both x and y? Or, in other words, How many points on the line have integral coördinates?



Since 3 and 15 have a common factor 3, it is clear that y is an integer if x is any multiple of 3. And so, when $x = 0, \pm 3, \pm 6, \pm 9$, etc., y is integral. You will see that these special points can be represented by the general form $(3n, 5 - 4n)$. This means that there is no end to the number of solutions of this equation in terms of integers, both positive and negative. But suppose there is some reason why the solutions must be not only integral but also positive. For instance, take this problem. A boy bought some apples at 3 cents each, and some oranges at 4 cents each and paid for all 15 cents. How many of each did he buy? Do you see that there is only one solution?

There are amusing problems leading to equations of this sort in which there are too few equations for a definite solution — one equation in two unknown quantities, or two equations in three unknown quantities, etc. — but owing to some restrictions on the answers, generally that they shall be integral, and often that they shall be positive too, the number of solutions is decidedly lessened. Sometimes the smallest positive integer that meets the conditions is asked for.

Before showing you how to undertake the search for integral solutions, it will be well to call to your attention some simple facts about the solution of these equations with their many answers. The fact that there is no one special solution, but that in general there are many solutions, accounts for the usual name of such equations — indeterminate equations.

In the equation which we were using, $4x + 3y = 15$, we noticed that the terms $3y$ and 15 are divisible by 3, and concluded that integral values of both x and y are possible only when $4x$ is also divisible by 3. But now suppose we have such an equation as this, $10x + 5y = 17$. It is impossible to satisfy the equation by integral values of x and y. For the first member is divisible by 5, and if 17 is equal to it, it should also be divisible by 5, and it is not. Let us show this in another way by solving for y.

$$y = \frac{17 - 10x}{5} = \frac{17}{5} - 2x,$$

and for every integral value of x, y is fractional. If the graph is drawn accurately, it will never pass through the points in which the ruled lines cut each other, for such points always represent pairs of integers. In general then, in the equation $ax + by = c$, there is no integral solution if a and b have a common factor not contained in c, but if a and b have no common factor, there will always be solutions in terms of integers.

Now I shall solve an indeterminate equation, using a method which you will find it easy to use on any equation of this kind. Let it be required to find integers which satisfy the equation $4x - 7y = 53$.

Solve for x, the unknown quantity with the smaller coefficient.

$$x = \frac{53 + 7y}{4} = 13 + y + \frac{1 + 3y}{4}.$$

If x and y are integers, $\frac{1 + 3y}{4}$ must be an integer; let it be represented by z.

Then $\frac{1 + 3y}{4} = z$, or $1 + 3y = 4z$.

As before, solve for the unknown quantity with the smaller coefficient.

$$y = \frac{4z - 1}{3} = z + \frac{z - 1}{3}.$$

Now let $\frac{z - 1}{3} = u$, then $z - 1 = 3u$, or $z = 3u + 1$.

Every integral value of u makes z an integer, and therefore x and y will be integers. Here are some of the many possible solutions. You will find it easy to check them.

$u =$	0	1	2	3	4	-1	-2
$z =$	1	4	7	10	·	-2	-5
$y =$	1	5	9	13	·	-3	-7
$x =$	15	22	29	36	·	8	1

Do you see that when u varies by 1 each time, z varies by $\cdots$, y by $\cdots$, x by $\cdots$? Evidently a general form for all integral solutions can be written, for if u is equal to n, the value of z is $3n + 1$, of y, $4n + 1$, and of x, $7n + 15$. For any integral value of n, positive or negative, integral values of x and y are found.

Of course it is often easy to see what value of z or u or some later number that you may have to introduce makes a fractional form equal to an integer. For instance, it is clear that $\frac{1 + 3y}{4}$ is integral when $y = 1$, and that $\frac{z - 1}{3}$ is integral when $z = 4$. But the regular method will always work, whether you happen to see the result in advance or not.

Suppose, for example, the form $\frac{3x + 5}{4}$ is to represent an integer. It is easy to see that $x = 1$ is a possible solution, that x must be odd, and that 5, 9, etc. are other admissible values of x.

In the regular method you will notice that the denominator decreases at each step, and therefore the coefficients in the numerator decrease, and that at last 1 is sure to appear.

PROBLEMS

The first of these problems are quite easy, some later ones are harder, and the last one will keep you occupied and amused for some time.

1. A man came into a postoffice and said, "I want some 2-cent stamps, 10 times as many 1-cent stamps, and the balance from this dollar bill in 5-cent stamps." How could his order be filled?

2. A man who wished to cash a $200 check asked for some 1-dollar bills, 10 times as many 2-dollar bills, and the balance in 5-dollar bills. How did the cashier carry out his request?

3. A jeweler bought 100 gems for $1000, paying $100 each for diamonds, $30 each for sapphires, and $5 each for turquoises. How many of each did he buy?

4. Find an integer of two digits which is 7 times the sum of its digits.

5. How many pairs of positive fractions are there whose denominators are 3 and 5, and whose sum is $2\frac{1}{15}$?

6. The numerator and denominator of a fraction consist of the same two digits but in opposite order, and the fraction is equal to $\frac{4}{7}$. Find the fraction.

7. A number consisting of three digits exceeds the sum of its digits by 180. If written in opposite order, it exceeds the sum of its digits by 378. Find the number.

8. Find all the pairs of unequal positive integers which have the property that the difference of their

squares is 5 times the difference of the numbers themselves.

9. Two numbers which differ by 54 are composed of the same two digits but in opposite order. Find the numbers.

10. A certain number, less than 200, is divided into four parts, such that, if the first is increased by 7, the second diminished by 6, the third multiplied by 5, and the fourth divided by 4, the results are all equal. Find the number.

11. A man emptied a barrel containing 126 quarts of oil, using a two-quart, a three-quart and a five-quart measure, filling each measure full each time it was used. He noticed that the amount that he drew with the three-quart measure was 5 times as much as what he drew with the two-quart measure. How much did he draw with each measure?

12. Find two integers, p and q, such that $31\,p + 73\,q = 1$.

Following the usual method,

$$p = \frac{1 - 73\,q}{31} = -\,2\,q + \frac{1 - 11\,q}{31},$$

$$\frac{1 - 11\,q}{31} = r, \quad q = \frac{1 - 31\,r}{11} = -\,2\,r + \frac{1 - 9\,r}{11}.$$

The coefficient of r in the numerator decreases more rapidly if instead of the last form we write

$$q = -\,3\,r + \frac{1 + 2\,r}{11},$$

and in general we can in such a case take our choice between writing a smaller quotient and having the similar term in the numerator with like sign, or a larger quotient with the similar term in the numerator with the opposite sign. It is better to choose that form which reduces the numbers in the fraction most rapidly, and so we will use this second form.

$$\frac{1 + 2r}{11} = s, \quad r = \frac{11s - 1}{2}$$

and any odd value of s will make r integral. Complete the solution.

Two pairs of values are $p = 33, q = -14$; $p = -40, q = 17$.

13. In a patriotic parade the boys and girls in all the schools of the town were invited to march, and each school was told to form its pupils in a column, with the same number in each row. In each school the number of pupils was less than 1000, and no two had the same number enrolled. Three schools had the following experience when the pupils were formed in a column: first they were arranged by twos, and there was one left over; then by threes, and there were two left over; then by fours, and there were three left over; then by fives, and there were four left over; then by sixes, and there were five left over; but finally, when they were arranged by sevens, it came out exactly right.

In the other schools, when the pupils were arranged two in a row, three in a row, etc., up through six in a row, there was one left over each time, but, as before, when seven were put in each row, none were left over. How many schools were there, and how many pupils in each school?

14. Five sailors had equal rights in a pile of cocoanuts. The night before they were to divide them one sailor, afraid of not getting his share, found that, if he gave one cocoanut to the pet monkey, the remainder was divisible by 5. He therefore threw one cocoanut to the monkey, and carried off his fifth. Soon afterward the second sailor decided to make sure of his share. He too gave one to the monkey, thus making the remainder divisible by 5, and carried off what he supposed was his fifth. This same thing was done by each one of the five sailors, each gave one cocoanut to the monkey and carried off one fifth of the remainder. In the morning, when they met to divide the cocoanuts, each looking as innocent as possible, the number in the pile was found to be divisible by 5. How many cocoanuts were there in the original pile?

You will find that these sailors must have been very strong men. The story might be a little more credible if the nuts were peanuts instead of cocoanuts.

You might like to try this puzzle first putting 2 or 3 or 4 wherever 5 occurs. The method is the same, but the work is shorter and therefore clearer.

CHAPTER XIII

DRAWING A STRAIGHT LINE WITHOUT A RULER

Have you ever noticed how differently you go about drawing a straight line and a circle?

If you are asked to draw a straight line, you look about for something with a straight edge which you can follow with a pencil — a ruler perhaps, or the cover of a book, or a folded sheet of paper.

But if you are asked to draw a circle, you do not look about for something with a circular edge which you can trace with a pencil — a coin for instance, or a box, or a saucer. If compasses are handy, you take them and draw a circle of any size you please. Or you may take a small slip of paper, pin one end of it to a sheet of paper, and make a hole in the other end for the pencil point. A very satisfactory circle may be drawn in this simple way.

Do you see how different these two methods are? In one case you insist upon having a straight line before you can construct a straight line, in the other case you can draw a circle without even looking at any other circle.

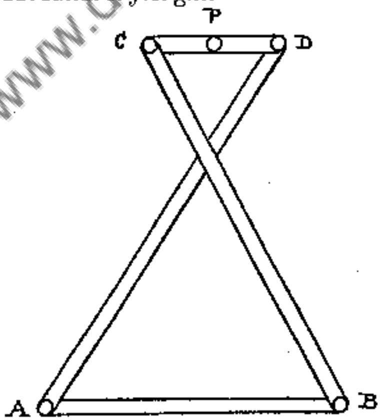
If you were asked in a Geometry class to construct an equilateral triangle, you would not think your solution of the problem very satisfactory if you found such a triangle somewhere and traced its outline. It is easy to see that you are not solving the problem geometrically.

Now suppose this problem is proposed: To draw a straight line without making any use of a straight line.

We have an apparatus for drawing a circle without making any use of a circle. Is there an apparatus for drawing a straight line without making any use of a straight line?

This question, though it may sound rather simple, yet proved very puzzling. One difference between drawing a straight line and a circle is that the circle may be drawn completely, but, since the line has no end, only a part of it can be drawn.

A Russian mathematician, named Tchebichev, invented a machine which drew a short line, approximately straight. You can easily make a rough model of his machine. Take four narrow strips of stiff pasteboard, tin, or any material which seems suitable. Two are to be of the same length, one shorter, and the fourth the shortest of all. Make a hole near each end of each strip so that the lengths of the strips, measured between the holes, shall be in the ratio 13 : 10 : 4. In the figure



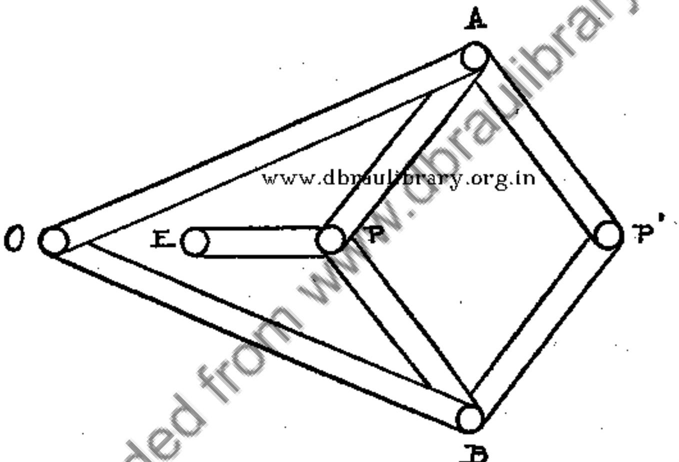
$$AD = BC \text{ and } AB : BC : CD = 10 : 13 : 4.$$

Join these pieces as shown in the figure, using paper fasteners or any attachments that will let the strips swing about. If now AB is held fixed on a sheet of paper and a pencil point is put through a hole at P, halfway between C and D, and is made to move, the pencil will draw a rather straight line.

This kind of device is called a linkage, because the pieces are linked together. Perhaps you think that we are really using straight lines in this linkage for drawing a straight line. I have used straight lines in the figure because they make it simpler, but it makes no difference at all how the edges of these strips are cut, they might be scalloped, or cut in any shapes. All that is necessary is to measure the distances between the holes as directed.

Soon after this machine was invented a French engineer, named Peaucellier, devised a linkage which draws a really straight line. This linkage, too, you can make easily. You need seven bars or strips with a hole in each end. In four of them the distances between the holes are the same, the fifth and sixth are longer, but the distances between the holes in these two are equal to each other, and the seventh I will describe later.

As the figure shows, the four equal rods are to be jointed together to form a rhombus, the two longer rods are to be jointed together and then attached to opposite corners of the rhombus so that all the rods can be swung easily. The point O is to be fixed, then you can see that A and B can move only on arcs of a circle about O as center, but P and P' can move much more freely. You will also find by experiment that if you move either P or P', the other one will be compelled to move. Now if P is made to move on an arc of a circle which passes through O, P' will describe a segment of a straight line. This can be accomplished by putting in another bar, the seventh, pivoted at the moving point P and a fixed point E, lying halfway between O and P, as indicated. P will then describe a circle with center at E, and if the machinery did not

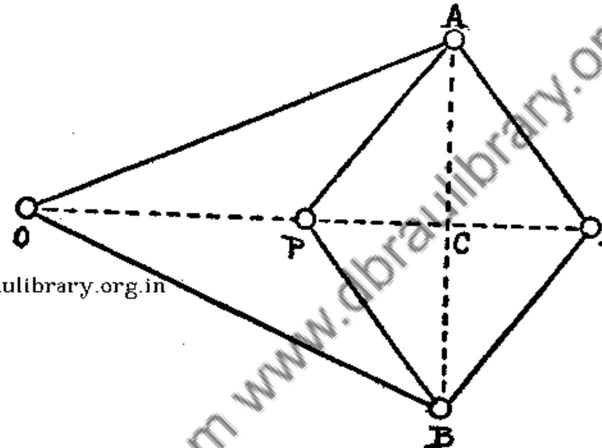


get in the way, this circle would pass through O. But it can swing about through only part of this distance, and P' will meanwhile describe a segment of a straight line.

These facts are of interest even if you do not care to learn the reasons why the line described by P' is straight. But the proof is not difficult, and so I shall put it in, in case anyone wishes to know why this very simple machine solves this really difficult problem.

I shall prove a series of statements leading finally to the desired proof.

1. In every position of the linkage $OP \cdot OP'$ is a constant.



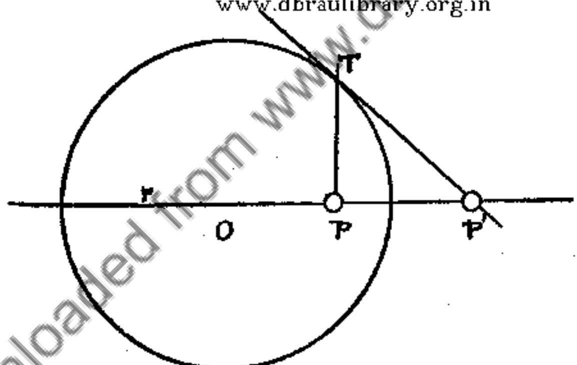
Draw the diagonals of the rhombus $APBP'$. They intersect at right angles. Call their point of intersection C. Then

$$\begin{aligned} OP \cdot OP' &= (OC - PC)(OC + CP') \\ &= (OC - PC)(OC + PC) \\ &= OC^2 - PC^2 \\ &= (OA^2 - AC^2) - (AP^2 - AC^2) \\ &= OA^2 - AP^2. \end{aligned}$$

Since OA and AP are constants, the difference of their squares is also a constant. That is, $OP \cdot OP'$ is a constant.

2. The constant found in the above proof is positive, since by construction OA is longer than AP. It is usually represented by r^2; r is evidently one side of a right triangle with hypotenuse OA and the remaining side AP. The length of r may also be found from the linkage by swinging A and P until OAP is a right triangle. You will find that P and P' will then coincide, and the rhombus is pulled out until two pairs of sides coincide. If a circle is drawn with center at O and radius r, the points P and P' are said to be inverse with respect to this circle.

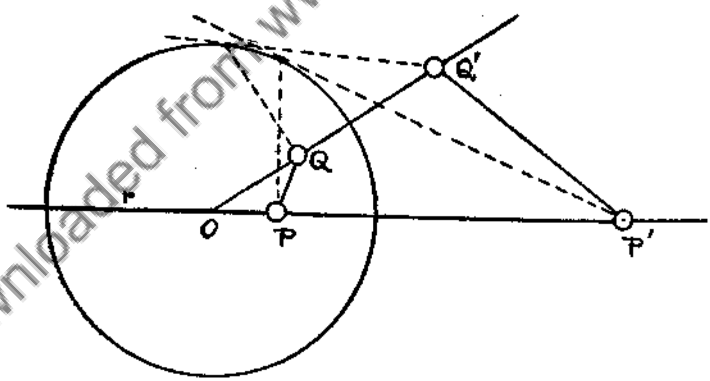
3. Problem. To any position of P to find its inverse point P'.



It is not necessary for the entire linkage to be drawn. For, wherever P may be, O, P, P' are in a straight line, and $OP \cdot OP'$ remains equal to a constant which we have called r^2, and we can easily construct r. Let us draw a circle with radius r and center O, and take any point P. If it is inside the circle, P' must be outside, for since $OP \cdot OP' = r^2$, if $OP < r$, $OP' > r$. Draw through P a perpendicular to OP, meeting the circle in T, and at T draw a tangent to the circle. The point in which this tangent meets OP is P'. For, by a familiar theorem about a right triangle, $OP : OT = OT : OP'$, that is, $OP \cdot OP' = r^2$. If the point chosen lies outside the circle, the figure looks just the same, but the tangent is drawn first from the outside point, and then the perpendicular to OP'.

The problem is therefore solved. You will find that if P is taken nearer to the circumference of the circle, P' also moves nearer to it. As P moves nearer to O, P' moves farther and farther away. What happens if P is taken on the left side of O?

4. Let P and Q be any two positions of one point, and P' and Q' be the corresponding inverse points.



The two angles OPQ and $OQ'P'$ will be shown to be equal.

The two triangles OPQ and $OQ'P'$ are similar since they have a common angle and the adjacent sides are proportional. For

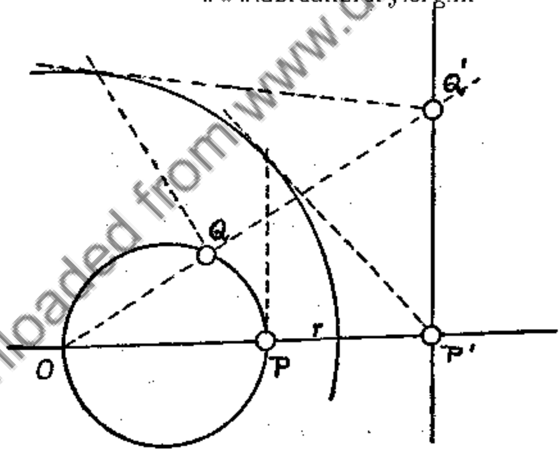
$$OP \cdot OP' = OQ \cdot OQ' = r^2.$$

Therefore $$OP : OQ = OQ' : OP'.$$

It follows that corresponding angles are equal; that is, $OPQ = OQ'P'$ and $OQP = OP'Q'$.

5. If P describes a circle passing through O, P' describes a straight line.

All the earlier statements are true for any position of P in the six-bar linkage. Now we shall add the seventh bar, and P will move on an arc of a circle which passes through O.



Let P have as its inverse P', and when P has moved to Q, let Q' be the inverse point. The question is: What path has the inverse point followed in going from P' to Q'?

Angle OQP is a right angle, therefore $OP'Q'$ is a right angle. This means that, if a point starting from P describes the circle PQO, and the corresponding point in each of its new positions is joined by a straight line to P', this line is perpendicular to OP'. But since at P' only one perpendicular to OP' can be drawn in this plane, the point Q' as it moves must always be on the perpendicular erected at P' to OP'; that is, as P describes a circle through O, P' describes a straight line.

It is therefore proved that this apparatus of Peaucellier draws a really straight line, not merely an approximation to one. Of course, in any machinery there may be slight errors which keep the drawing from being absolutely exact, but the more perfect the machine is the more accurate the drawing will be, and if it could be quite perfect, so would the line be which the point describes. In the machine which is known as an inversor or a Peaucellier cell a pencil is attached at P' so that the line is really drawn. You can easily make an inversor which will work very well.

CHAPTER XIV

THE IMPOSSIBLE IN MATHEMATICS

In work in Mathematics, and indeed in any subject, we occasionally run across something which we cannot do, try as we may. Sometimes we find that later, after our minds have had a chance to rest a little, or perhaps to grow more familiar with the problem, we come back to it and solve it without much difficulty. Sometimes a hint from a wiser person helps us over the hard place.

But are there any problems that really cannot be solved? It is evidently one thing to say that a problem has never been solved, and quite another thing to say that it cannot be solved.

It is very easy to find some things in Mathematics which cannot be done. For instance, the following constructions are impossible.

1. To draw a five-sided square.
2. From a point without a line to draw two perpendiculars to the line.

Each of these calls for something which contradicts our previous knowledge. One contradicts the definition of a square, and we must never go against a definition. The other contradicts a theorem already established. The indirect method, or *reductio ad absurdum,* is based upon the principle that facts already proved must not be contradicted.

Another thing which is impossible is to divide by zero. As one writer puts it, "The first law of Mathematics is *Thou shalt not divide by zero.*" All sorts of absurdities arise if we try to treat zero in dividing as we would treat any other number. For instance,

$$3 \times 0 = 11 \times 0.$$

Therefore, removing a common factor,

$$3 = 11.$$

Many of the amusing "proofs" of untruths, such as that $1 = 2$, that a fly weighs as much as an elephant, etc., depend upon division by zero to bring about the absurd result.

As another illustration of the impossible take the proof in Chapter VIII that it is impossible to square a rational number and get 2 as the result.

Another illustration of the same kind is found in Chapter IX, where it was stated that π is not a rational number; that is, it is impossible to write a fraction which gives exactly the value of π.

You know that you can get quite elaborate irrational numbers if you combine by addition, division, etc., all kinds of roots of integers; for example,

$$\frac{\sqrt{17 + \sqrt[4]{23}}}{\sqrt[5]{479} - \sqrt[8]{826}} - \sqrt[3]{7 - \sqrt[5]{41}}.$$

But every such combination, using any roots you please, forming fractions, adding, etc., provided you use only a finite number of operations, can always be the root of an algebraic equation, that is, an equation of the form

$$ax^n + bx^{n-1} + cx^{n-2} + \cdots + rx + s = 0,$$

where every letter except x stands for an integer, and n is positive.

Now in the last century another fact about π was proved. No matter how complicated algebraic equations you may experiment with, π can never be one of the roots. That is, π is not only not a rational number, it is not even an algebraic number. It is what is known as a transcendental number. This sounds rather awe inspiring, but you may have met already some other members of the same family, for most of our sines, cosines, and logarithms are transcendental numbers. Some years hence you may know more about them. For the present it is interesting to know that there are such numbers, and it helps one to see how foolish and ignorant those men are who are still trying to find an exact value for π. The reason why they still continue this useless search is that they do not realize the difference between "It has not yet been solved" and "It cannot be solved." And when a mathematician has proved that a thing cannot be done, it is well to stop trying to do it, and to spend our energy on something that we perhaps can do.

Still another kind of impossibility is that of solving a problem with inadequate means.

Think, for example, of a trip to the moon. We are rather cautious nowadays about claiming that such a feat can never be accomplished, but we may safely say that a trip to the moon cannot be made in the *Los Angeles* or any kind of flying machine now in existence.

A misunderstanding of the conditions of a problem has been responsible for many foolish claims made by persons untrained in Mathematics. For instance, in many newspapers this past summer there were great headlines of this kind: A problem that has baffled mathematicians since Euclid's day is solved!! The angle is at last trisected!!

Now anyone who knows anything about the subject can tell you that the angle was trisected long ago, and that there are many means of accomplishing this feat. As long ago as 420 B.C. a geometer named Hippias invented a curve called the quadratrix by which an angle can be divided in any ratio. He also invented an instrument which drew this curve. Another curve used about 180 B.C. for the same purpose was the conchoid of Nicomedes. Other curves which enable one to trisect an angle are the equilateral hyperbola and the trisectrix.

The reason why this problem proved so difficult for the old geometers, and so gained a reputation for itself, was that they tried to solve it by the use of ruler and compasses only, the use of other methods being frowned upon by Euclid and other Greek geometers. To this day the name "Euclidean construction" is given to those constructions only which are confined to the use of ruler and compasses. Of course you know that by the methods of elementary Geometry you can trisect a right angle, but it has been proved that a general angle cannot be trisected if one's instruments are ruler and compasses only. All constructions by these elementary means give at best only approximate solutions.

Another problem which cannot be solved by ruler and compasses only is the one known as the duplication of the cube. It calls for the edge of a cube which shall have exactly twice the volume of a given cube. This problem, for some reason, has proved less attractive to valiant but ignorant would-be solvers than the trisection of an angle or the squaring of the circle, and I do not remember any announcements of its solution in any newspapers.

Let us turn now to problems of the other variety, those not yet solved, but hopefully attacked by mathematicians. Here are some of them, one interesting thing about which is that you can see clearly what they mean. You have had many problems given you to solve, the meaning of which was much harder to grasp.

1. Is there any end to the number of twin primes?

In Chapter I, I told you that there are fewer twin primes as one goes farther out, at least as far as to 1000. The question has been investigated as far out as 8,100,000, and twin primes are found to occur more and more rarely. But no one knows whether they ever come to an end, whether there is a point at which we may say, "That is the largest pair of twin primes."

2. Are there any perfect numbers besides the nine already found? Is there such a thing as an odd perfect number? Does every even perfect number end in 6 or 28, as is the case with those now known?

3. Is it true that every even integer is the sum of two prime numbers? For instance, $18 = 7 + 11$, $50 = 19 + 31$.

This statement, known as Goldbach's Theorem, is true in every case that has been tested, but it has not yet been proved true.

4. Are there any positive integral values of n, x, y, z which satisfy the equation $x^n + y^n = z^n$ when n is greater than 2?

We know that when $n = 2$, there is no end to the possible values of x, y, and z. See Chapter IV. But it has been proved that when $n = 3, 4, 5$, etc., at least as far as 7000, there are no positive integral values of x, y, z which satisfy the equation. That is, the sum of two cubes is never a cube, the sum of two fourth powers is never a fourth power, etc. Of course it is strongly suspected that the answer to this question is *No*, but it is not yet proved.

About three centuries ago a French mathematician, named Fermat, announced that he had proved that there is no solution when n is greater than 2, but he did not give his proof, and it still remains a famous unsolved problem. A large sum of money has been offered as a prize to the first one who gives a complete answer to this question before the year 2007.

I have tried to show you in this chapter some mathematical impossibilities. Anyone who knows even a little about them is not likely to join the army of circle-squarers or angle-trisectors. There is one fine thing to be said about these useless efforts. They are often due to a noble characteristic of human nature, the kind of ambition which sends men to explore dangerous lands or to climb forbidding mountains, and which made men try for centuries to learn to fly. Just because these are hard tests of strength and skill they prove attractive. So when men get the idea that mathematicians have not yet managed to square the circle, that is, to find an exact or rational value for its area, simply because it has proved too hard for them to do, they long to solve the problem, and so to beat all the mathematicians of the ages. The trouble is that they do not know enough even to understand that it cannot be done.

There is no way of proving that Mt. Everest cannot be climbed to the very top, and explorers will continue to make the attempt, and some day the summit may be reached. But no one will ever find an exact rational value of π, and the work which is being carried on in this search is only wasted effort. Only last year a book was sent to me which claims to prove that $\pi = \frac{256}{81}$, the value found in an Egyptian manuscript of about 1700 B.C. The author of this book, so far as I can judge, knows little Algebra and less Geometry, and has no idea what a proof is. Very often the less we know the more cocksure we are.

In Histories of Mathematics you will find readable accounts of attempts made in ancient times to solve some famous problems, and also old stories in which the Greeks told of their mathematical difficulties in a very interesting form; as, for instance, the story of the duplication of the cube.

ANSWERS TO SOME OF THE PROBLEMS

Chapter	Page	Number	
IV	42	14	7,744
		15	4,225
		16	34,596
		17	45,796
		18	455,625
		20	1,089
V	60	8	615,384

XI 109 7 Let the four numbers be x, y, z, u, their sum k, and let c be the number to be added, subtracted, etc., and a the common result. Then

$$x + y + z + u = k,$$
$$x + c = y - c = cz = \frac{u}{c} = a.$$

These may be written

$$x = a - c,$$
$$y = a + c,$$
$$z = \frac{a}{c},$$
$$u = ac.$$

By addition,

$$x + y + z + u = \frac{a}{c}(c + 1)^2 = k.$$

For any specific value of k the corresponding values of a and c may be found.

Chapter	Page	Number	
	116	33	The larger is 5776.
		34	1681.
	117	35	One number is 45.
XII	124	5	Two.
		6	$\frac{12}{21}$, $\frac{24}{42}$, $\frac{36}{63}$, $\frac{48}{84}$.
		7	193
		8	1 and 4; 2 and 3.

CHAPTER	PAGE	NUMBER	
	125	10	Five possible values, the smallest 61, the largest 185.
		11	6, 30, 90.
	126	13	Number of schools 5; number of pupils 119, 301, 539, 721, 959.
	127	14	If 2 is used, the number is 11 or 19 or · · · If 3 is used, the number is 25 or 106 or · · · If 4 is used, the number is 765 or 1789 or · · · If 5 is used, the number is 3121 or 18,746 or · · · If 6 is used, the number is 233,275 or 513,211 or · · ·

Catalog of DOVER BOOKS

BOOKS EXPLAINING SCIENCE

(Note: The books listed under this category are general introductions, surveys, reviews, and non-technical expositions of science for the interested layman or scientist who wishes to brush up. Dover also publishes the largest list of inexpensive reprints of books on intermediate and higher mathematics, mathematical physics, engineering, chemistry, astronomy, etc., for the professional mathematician or scientist. For our complete Science Catalog, write Dept. catrr., Dover Publications, Inc., 180 Varick Street, New York 14, N. Y.)

CONCERNING THE NATURE OF THINGS, Sir William Bragg. Royal Institute Christmas Lectures by Nobel Laureate. Excellent plain-language introduction to gases, molecules, crystal structure, etc. explains "building blocks" of universe, basic properties of matter, with simplest, clearest examples, demonstrations. 32pp. of photos; 57 figures. 244pp. 5⅜ x 8.
T31 Paperbound **$1.35**

MATTER AND LIGHT, THE NEW PHYSICS, Louis de Broglie. Non-technical explanations by a Nobel Laureate of electro-magnetic theory, relativity, wave mechanics, quantum physics, philosophies of science, etc. Simple, yet accurate introduction to work of Planck, Bohr, Einstein, other modern physicists. Only 2 of 12 chapters require mathematics. 300pp. 5⅜ x 8.
T35 Paperbound **$1.60**

THE COMMON SENSE OF THE EXACT SCIENCES, W. K. Clifford. For 70 years, Clifford's work has been acclaimed as one of the clearest, yet most precise introductions to mathematical symbolism, measurement, surface boundaries, position, space, motion, mass and force, etc. Prefaces by Bertrand Russell and Karl Pearson. Introduction by James Newman. 130 figures. 249pp. 5⅜ x 8.
T61 Paperbound **$1.60**

THE NATURE OF LIGHT AND COLOUR IN THE OPEN AIR, M. Minnaert. What causes mirages? haloes? "multiple" suns and moons? Professor Minnaert explains these and hundreds of other fascinating natural optical phenomena in simple terms, tells how to observe them, suggests hundreds of experiments. 200 illus; 42 photos. xvi + 362pp.
T196 Paperbound **$1.95**

SPINNING TOPS AND GYROSCOPIC MOTION, John Perry. Classic elementary text on dynamics of rotation treats gyroscopes, tops, how quasi-rigidity is induced in paper disks, smoke rings, chains, etc, by rapid motion, precession, earth's motion, etc. Contains many easy-to-perform experiments. Appendix on practical uses of gyroscopes. 62 figures. 128pp.
T416 Paperbound **$1.00**

A CONCISE HISTORY OF MATHEMATICS, D. Struik. This lucid, easily followed history of mathematics from the Ancient Near East to modern times requires no mathematical background itself, yet introduces both mathematicians and laymen to basic concepts and discoveries and the men who made them. Contains a collection of 31 portraits of eminent mathematicians. Bibliography. xix + 299pp. 5⅜ x 8.
T255 Paperbound **$1.75**

THE RESTLESS UNIVERSE, Max Born. A remarkably clear, thorough exposition of gases, electrons, ions, waves and particles, electronic structure of the atom, nuclear physics, written for the layman by a Nobel Laureate. "Much more thorough and deep than most attempts . . . easy and delightful," CHEMICAL AND ENGINEERING NEWS. Includes 7 animated sequences showing motion of molecules, alpha particles, etc. 11 full-page plates of photographs. Total of nearly 600 illus. 315pp. 6⅛ x 9¼.
T412 Paperbound **$2.00**

WHAT IS SCIENCE?, N. Campbell. The role of experiment, the function of mathematics, the nature of scientific laws, the limitations of science, and many other provocative topics are explored without technicalities by an eminent scientist. "Still an excellent introduction to scientific philosophy," H. Margenau in PHYSICS TODAY. 192pp. 5⅜ x 8.
S43 Paperbound **$1.25**

FADS AND FALLACIES IN THE NAME OF SCIENCE, Martin Gardner. The standard account of the various cults, quack systems and delusions which have recently masqueraded as science: hollow earth theory, Atlantis, dianetics, Reich's orgone theory, flying saucers, Bridey Murphy, psionics, irridiagnosis, many other fascinating fallacies that deluded tens of thousands. "Should be read by everyone, scientist and non-scientist alike," R. T. Birge, Prof. Emeritus, Univ. of California; Former President, American Physical Society. Formerly titled, "In the Name of Science." Revised and enlarged edition. x + 365pp. 5⅜ x 8.
T394 Paperbound $1.50

THE STUDY OF THE HISTORY OF MATHEMATICS, THE STUDY OF THE HISTORY OF SCIENCE, G. Sarton. Two books bound as one. Both volumes are standard introductions to their fields by an eminent science historian. They discuss problems of historical research, teaching, pitfalls, other matters of interest to the historically oriented writer, teacher, or student. Both have extensive bibliographies. 10 illustrations. 188pp. 5⅜ x 8. T240 Paperbound $1.25

THE PRINCIPLES OF SCIENCE, W. S. Jevons. Unabridged reprinting of a milestone in the development of symbolic logic and other subjects concerning scientific methodology, probability, inferential validity, etc. Also describes Jevons' "logic machine," an early precursor of modern electronic calculators. Preface by E. Nagel. 839pp. 5⅜ x 8. S446 Paperbound $2.98

SCIENCE THEORY AND MAN, Erwin Schroedinger. Complete, unabridged reprinting of "Science and the Human Temperament" plus an additional essay "What is an Elementary Particle?" Nobel Laureate Schroedinger discusses many aspects of modern physics from novel points of view which provide unusual insights for both laymen and physicists. 192 pp. 5⅜ x 8.
T428 Paperbound $1.35

BRIDGES AND THEIR BUILDERS, D. B. Steinman & S. R. Watson. Information about ancient, medieval, modern bridges; how they were built; who built them; the structural principles employed; the materials they are built of; etc. Written by one of the world's leading authorities on bridge design and construction. New, revised, expanded edition. 23 photos; 26 line drawings, xvii + 401pp. 5⅜ x 8. T431 Paperbound $1.95

HISTORY OF MATHEMATICS, D. E. Smith. Most comprehensive non-technical history of math in English. In two volumes. Vol. I: A chronological examination of the growth of mathematics from primitive concepts up to 1900. Vol. II: The development of ideas in specific fields and areas, up through elementary calculus. The lives and works of over a thousand mathematicians are covered; thousands of specific historical problems and their solutions are clearly explained. Total of 510 illustrations, 1355pp. 5⅜ x 8. Set boxed in attractive container.
T429, T430 Paperbound, the set $5.00

PHILOSOPHY AND THE PHYSICISTS, L. S. Stebbing. A philosopher examines the philosophical implications of modern science by posing a lively critical attack on the popular science expositions of Sir James Jeans and Arthur Eddington. xvi + 295pp. 5⅜ x 8.
T480 Paperbound $1.65

ON MATHEMATICS AND MATHEMATICIANS, R. E. Moritz. The first collection of quotations by and about mathematicians in English. 1140 anecdotes, aphorisms, definitions, speculations, etc. give both mathematicians and layman stimulating new insights into what mathematics is, and into the personalities of the great mathematicians from Archimedes to Euler, Gauss, Klein, Weierstrass. Invaluable to teachers, writers. Extensive cross index. 410pp. 5⅜ x 8.
T489 Paperbound $1.95

NATURAL SCIENCE, BIOLOGY, GEOLOGY, TRAVEL

A SHORT HISTORY OF ANATOMY AND PHYSIOLOGY FROM THE GREEKS TO HARVEY, C. Singer. A great medical historian's fascinating intermediate account of the slow advance of anatomical and physiological knowledge from pre-scientific times to Vesalius, Harvey. 139 unusually interesting illustrations. 221pp. 5⅜ x 8. T389 Paperbound $1.75

THE BEHAVIOUR AND SOCIAL LIFE OF HONEYBEES, Ronald Ribbands. The most comprehensive, lucid and authoritative book on bee habits, communication, duties, cell life, motivations, etc. "A MUST for every scientist, experimenter, and educator, and a happy and valuable selection for all interested in the honeybee," AMERICAN BEE JOURNAL. 690-item bibliography. 127 illus.; 11 photographic plates. 352pp. 5⅜ x 8⅜. S410 Clothbound $4.50

TRAVELS OF WILLIAM BARTRAM, edited by **Mark Van Doren.** One of the 18th century's most delightful books, and one of the few first-hand sources of information about American geography, natural history, and anthropology of American Indian tribes of the time. "The mind of a scientist with the soul of a poet," John Livingston Lowes. 13 original illustrations, maps. Introduction by Mark Van Doren. 448pp. 5⅜ x 8. T326 Paperbound $2.00

STUDIES ON THE STRUCTURE AND DEVELOPMENT OF VERTEBRATES, Edwin Goodrich. The definitive study of the skeleton, fins and limbs, head region, divisions of the body cavity, vascular, respiratory, excretory systems, etc., of vertebrates from fish to higher mammals, by the greatest comparative anatomist of recent times. "The standard textbook," JOURNAL OF ANATOMY. 754 illus. 69-page biographical study. 1186-item bibliography. 2 vols. Total of 906pp. 5⅜ x 8.
Vol. I: S449 Paperbound $2.50
Vol. II: S450 Paperbound $2.50

THE BIRTH AND DEVELOPMENT OF THE GEOLOGICAL SCIENCES, F. D. Adams. The most complete and thorough history of the earth sciences in print. Covers over 300 geological thinkers and systems; treats fossils, theories of stone growth, paleontology, earthquakes, vulcanists vs. neptunists, odd theories, etc. 91 illustrations, including medieval, Renaissance wood cuts, etc. 632 footnotes and bibliographic notes. 511pp. 308pp. 5⅜ x 8. T5 Paperbound **$2.00**

FROM MAGIC TO SCIENCE, Charles Singer. A close study of aspects of medical science from the Roman Empire through the Renaissance. The sections on early herbals, and "The Visions of Hildegarde of Bingen," are probably the best studies of these subjects available. 158 unusual classic and medieval illustrations. xxvii + 365pp. 5⅜ x 8. T390 Paperbound **$2.00**

SAILING ALONE AROUND THE WORLD, Captain Joshua Slocum. Captain Slocum's personal account of his single-handed voyage around the world in a 34-foot boat he rebuilt himself. A classic of both seamanship and descriptive writing. "A nautical equivalent of Thoreau's account," Van Wyck Brooks. 67 illus. 308pp. 5⅜ x 8. T326 Paperbound **$1.00**

TREES OF THE EASTERN AND CENTRAL UNITED STATES AND CANADA, W. M. Harlow. Standard middle-level guide designed to help you know the characteristics of Eastern trees and identify them at sight by means of an 8-page synoptic key. More than 600 drawings and photographs of twigs, leaves, fruit, other features. xiii + 288pp. 4⅝ x 6½. T395 Paperbound **$1.35**

FRUIT KEY AND TWIG KEY ("Fruit Key to Northeastern Trees," "Twig Key to Deciduous Woody Plants of Eastern North America"), **W. M. Harlow.** Identify trees in fall, winter, spring. Easy-to-use, synoptic keys, with photographs of every twig and fruit identified. Covers 120 different fruits, 160 different twigs. Over 350 photos. Bibliographies. Glossaries. Total of 143pp. 5⅝ x 8⅜. T511 Paperbound **$1.25**

INTRODUCTION TO THE STUDY OF EXPERIMENTAL MEDICINE, Claude Bernard. This classic records Bernard's far-reaching efforts to transform physiology into an exact science. It covers problems of vivisection, the limits of physiological experiment, hypotheses in medical experimentation, hundreds of others. Many of his own famous experiments on the liver, the pancreas, etc., are used as examples. Foreword by I. B. Cohen. xxviii + 266pp. 5⅜ x 8. T400 Paperbound **$1.50**

THE ORIGIN OF LIFE, A. I. Oparin. The first modern statement that life evolved from complex nitro-carbon compounds, carefully presented according to modern biochemical knowledge of primary colloids, organic molecules, etc. Begins with historical introduction to the problem of the origin of life. Bibliography. xxv + 270pp. 5⅜ x 8. S213 Paperbound **$1.75**

A HISTORY OF ASTRONOMY FROM THALES TO KEPLER, J. L. E. Dreyer. The only work in English which provides a detailed picture of man's cosmological views from Egypt, Babylonia, Greece, and Alexandria to Copernicus, Tycho Brahe and Kepler. "Standard reference on Greek astronomy and the Copernican revolution," SKY AND TELESCOPE. Formerly called "A History of Planetary Systems From Thales to Kepler." Bibliography. 21 diagrams. xvii + 430pp. 5⅜ x 8. S79 Paperbound **$1.98**

URANIUM PROSPECTING, H. L. Barnes. A professional geologist tells you what you need to know. Hundreds of facts about minerals, tests, detectors, sampling, assays, claiming, developing, government regulations, etc. Glossary of technical terms. Annotated bibliography. x + 117pp. 5⅜ x 8. T309 Paperbound **$1.00**

DE RE METALLICA, Georgius Agricola. All 12 books of this 400 year old classic on metals and metal production, fully annotated, and containing all 289 of the 16th century woodcuts which made the original an artistic masterpiece. A superb gift for geologists, engineers, libraries, artists, historians. Translated by Herbert Hoover & L. H. Hoover. Bibliography, survey of ancient authors. 289 illustrations of the excavating, assaying, smelting, refining, and countless other metal production operations described in the text. 672pp. 6¾ x 10¾. Deluxe library edition. S6 Clothbound **$10.00**

DE MAGNETE, William Gilbert. A landmark of science by the man who first used the word "electricity," distinguished between static electricity and magnetism, and founded a new science. P. F. Mottelay translation. 90 figures. lix + 368pp. 5⅜ x 8. S470 Paperbound **$2.00**

THE AUTOBIOGRAPHY OF CHARLES DARWIN AND SELECTED LETTERS, Francis Darwin, ed. Fascinating documents on Darwin's early life, the voyage of the "Beagle," the discovery of evolution, Darwin's thought on mimicry, plant development, vivisection, evolution, many other subjects Letters to Henslow, Lyell, Hooker, Wallace, Kingsley, etc. Appendix. 365pp. 5⅜ x 8. T479 Paperbound **$1.65**

A WAY OF LIFE AND OTHER SELECTED WRITINGS OF SIR WILLIAM OSLER. 16 of the great physician, teacher and humanist's most inspiring writings on a practical philosophy of life, science and the humanities, and the history of medicine. 5 photographs. Introduction by G. L. Keynes, M.D., F.R.C.S. xx + 278pp. 5⅜ x 8. T488 Paperbound **$1.50**

LITERATURE

WORLD DRAMA, B. H. Clark. 46 plays from Ancient Greece, Rome, to India, China, Japan. Plays by Aeschylus, Sophocles, Euripides, Aristophanes, Plautus, Marlowe, Jonson, Farquhar, Goldsmith, Cervantes, Molière, Dumas, Goethe, Schiller, Ibsen, many others. One of the most comprehensive collections of important plays from all literature available in English. Over ⅓ of this material is unavailable in any other current edition. Reading lists. 2 volumes. Total of 1364pp. 5⅜ x 8. Vol. I, T57 Paperbound **$2.00**
Vol. II, T59 Paperbound **$2.00**

MASTERS OF THE DRAMA, John Gassner. The most comprehensive history of the drama in print. Covers more than 800 dramatists and over 2000 plays from the Greeks to modern Western, Near Eastern, Oriental drama. Plot summaries, theatre history, etc. "Best of its kind in English," NEW REPUBLIC. 35 pages of bibliography. 77 photos and drawings. Deluxe edition. xxii + 890pp. 5⅜ x 8. T100 Clothbound **$5.95**

THE DRAMA OF LUIGI PIRANDELLO, D. Vittorini. All 38 of Pirandello's plays (to 1935) summarized and analyzed in terms of symbolic techniques, plot structure, etc. The only authorized work. Foreword by Pirandello. Biography. Bibliography. xiii + 350pp. 5⅜ x 8. T435 Paperbound **$1.98**

ARISTOTLE'S THEORY OF POETRY AND THE FINE ARTS, S. H. Butcher, ed. The celebrated "Butcher translation" faced page by page with the Greek text; Butcher's 300-page introduction to Greek poetic, dramatic thought. Modern Aristotelian criticism discussed by John Gassner. lxxvi + 421pp. 5⅜ x 8. T42 Paperbound **$2.00**

EUGENE O'NEILL: THE MAN AND HIS PLAYS, B. H. Clark. The first published source-book on O'Neill's life and work. Analyzes each play from the early THE WEB up to THE ICEMAN COMETH. Supplies much information about environmental and dramatic influences. ix + 182pp. 5⅜ x 8. T379 Paperbound **$1.25**

INTRODUCTION TO ENGLISH LITERATURE, B. Dobrée, ed. Most compendious literary aid in its price range. Extensive, categorized bibliography (with entries up to 1949) of more than 5,000 poets, dramatists, novelists, as well as historians, philosophers, economists, religious writers, travellers, and scientists of literary stature. Information about manuscripts, important biographical data. Critical, historical, background works not simply listed, but evaluated. Each volume also contains a long introduction to the period it covers.

Vol. I: **THE BEGINNINGS OF ENGLISH LITERATURE TO SKELTON, 1509, W. L. Renwick. H. Orton.** 450pp. 5⅛ x 7⅛. T75 Clothbound **$3.50**

Vol. II: **THE ENGLISH RENAISSANCE, 1510-1688, V. de Sola Pinto.** 381pp. 5⅛ x 7⅛. T76 Clothbound **$3.50**

Vol. III: **THE AUGUSTANS AND ROMANTICS, 1689-1830, H. Dyson, J. Butt.** 320pp. 5⅛ x 7⅛. T77 Clothbound **$3.50**

Vol. IV: **THE VICTORIANS AND AFTER, 1830-1914, E. Batho, B. Dobrée.** 360pp. 5⅛ x 7⅛. T78 Clothbound **$3.50**

EPIC AND ROMANCE, W. P. Ker. The standard survey of Medieval epic and romance by a foremost authority on Medieval literature. Covers historical background, plot, literary analysis, significance of Teutonic epics, Icelandic sagas, Beowulf, French chansons de geste, the Niebelungenlied, Arthurian romances, much more. 422pp. 5⅜ x 8. T355 Paperbound **$1.95**

THE HEART OF EMERSON'S JOURNALS, Bliss Perry, ed. Emerson's most intimate thoughts, impressions, records of conversations with Channing, Hawthorne, Thoreau, etc., carefully chosen from the 10 volumes of The Journals. "The essays do not reveal the power of Emerson's mind . . .as do these hasty and informal writings," N. Y. TIMES. Preface by B. Perry. 370pp. 5⅜ x 8. T447 Paperbound **$1.85**

A SOURCE BOOK IN THEATRICAL HISTORY, A. M. Nagler. (Formerly, "Sources of Theatrical History.") Over 300 selected passages by contemporary observers tell about styles of acting, direction, make-up, scene designing, etc., in the theatre's great periods from ancient Greece to the Théâtre Libre. "Indispensable complement to the study of drama," EDUCATIONAL THEATRE JOURNAL. Prof. Nagler, Yale Univ. School of Drama, also supplies notes, references. 85 illustrations. 611pp. 5⅜ x 8. T515 Paperbound **$2.75**

THE ART OF THE STORY-TELLER, M. L. Shedlock. Regarded as the finest, most helpful book on telling stories to children, by a great story-teller. How to catch, hold, recapture attention; how to choose material; many other aspects. Also includes: a 99-page selection of Miss Shedlock's most successful stories; extensive bibliography of other stories. xxi + 320pp. 5⅜ x 8. T245 Clothbound **$3.50**

THE DEVIL'S DICTIONARY, Ambrose Bierce. Over 1000 short, ironic definitions in alphabetical order, by America's greatest satirist in the classical tradition. "Some of the most gorgeous witticisms in the English language," H. L. Mencken. 144pp. 5⅜ x 8. T487 Paperbound **$1.00**

MUSIC

A DICTIONARY OF HYMNOLOGY, John Julian. More than 30,000 entries on individual hymns, their authorship, textual variations, location of texts, dates and circumstances of composition, denominational and ritual usages, the biographies of more than 9,000 hymn writers, essays on important topics such as children's hymns and Christmas carols, and hundreds of thousands of other important facts about hymns which are virtually impossible to find anywhere else. Convenient alphabetical listing, and a 200-page double-columned index of first lines enable you to track down virtually any hymn ever written. Total of 1786pp. 6¼ x 9¼. 2 volumes. T133. The Set, Clothbound **$15.00**

STRUCTURAL HEARING, TONAL COHERENCE IN MUSIC, Felix Salzer. Extends the well-known Schenker approach to include modern music, music of the middle ages, and Renaissance music. Explores the phenomenon of tonal organization by discussing more than 500 compositions, and offers unusual new insights into the theory of composition and musical relationships. "The foundation on which all teaching in music theory has been based at this college," Leopold Mannes, President, The Mannes College of Music. Total of 658pp. 6½ x 9¼. 2 volumes. S418 The set, Clothbound **$8.00**

A GENERAL HISTORY OF MUSIC, Charles Burney. The complete history of music from the Greeks up to 1789 by the 18th century musical historian who personally knew the great Baroque composers. Covers sacred and secular, vocal and instrumental, operatic and symphonic music; treats theory, notation, forms, instruments; discusses composers, performers, important works. Invaluable as a source of information on the period for students, historians, musicians. "Surprisingly few of Burney's statements have been invalidated by modern research . . . still of great value," NEW YORK TIMES. Edited and corrected by Frank Mercer. 35 figures. 1915pp. 5½ x 8½. 2 volumes. T36 The set, Clothbound **$12.50**

JOHANN SEBASTIAN BACH, Phillip Spitta. Recognized as one of the greatest accomplishments of musical scholarship and far and away the definitive coverage of Bach's works. Hundreds of individual pieces are analyzed. Major works, such as the B Minor Mass and the St. Matthew Passion are examined in minute detail. Spitta also deals with the works of Buxtehude, Pachelbel, and others of the period. Can be read with profit even by those without a knowledge of the technicalities of musical composition. "Unchallenged as the last word on one of the supreme geniuses of music," John Barkham, SATURDAY REVIEW SYNDICATE. Total of 1819pp. 5⅜ x 8. 2 volumes. T252 The set, Clothbound **$10.00**

HISTORY

THE IDEA OF PROGRESS, J. B. Bury. Prof. Bury traces the evolution of a central concept of Western civilization in Greek, Roman, Medieval, and Renaissance thought to its flowering in the 17th and 18th centuries. Introduction by Charles Beard. xl + 357pp. 5⅜ x 8.
T39 Clothbound **$3.95**
T40 Paperbound **$1.95**

THE ANCIENT GREEK HISTORIANS, J. B. Bury. Greek historians such as Herodotus, Thucydides, Xenophon; Roman historians such as Tacitus, Caesar, Livy; scores of others fully analyzed in terms of sources, concepts, influences, etc., by a great scholar and historian. 291pp. 5⅜ x 8. T397 Paperbound **$1.50**

HISTORY OF THE LATER ROMAN EMPIRE, J. B. Bury. The standard work on the Byzantine Empire from 395 A.D. to the death of Justinian in 565 A.D., by the leading Byzantine scholar of our time. Covers political, social, cultural, theological, military history. Quotes contemporary documents extensively. "Most unlikely that it will ever be superseded," Glanville Downey, Dumbarton Oaks Research Library. Genealogical tables. 5 maps. Bibliography. 2 vols. Total of 965pp. 5⅜ x 8. T398, T399 Paperbound, the set **$4.00**

GARDNER'S PHOTOGRAPHIC SKETCH BOOK OF THE CIVIL WAR, Alexander Gardner. One of the rarest and most valuable Civil War photographic collections exactly reproduced for the first time since 1866. Scenes of Manassas, Bull Run, Harper's Ferry, Appomattox, Mechanicsville, Fredericksburg, Gettysburg, etc.; battle ruins, prisons, arsenals, a slave pen, fortifications; Lincoln on the field, officers, men, corpses. By one of the most famous pioneers in documentary photography. Original copies of the "Sketch Book" sold for $425 in 1952. Introduction by E. Bleiler. 100 full-page 7 x 10 photographs (original size). 244pp. 10¾ x 8½. T476 Clothbound **$6.00**

THE WORLD'S GREAT SPEECHES, L. Copeland and L. Lamm, eds. 255 speeches from Pericles to Churchill, Dylan Thomas. Invaluable as a guide to speakers; fascinating as history past and present; a source of much difficult-to-find material. Includes an extensive section of informal and humorous speeches. 3 indices: Topic, Author, Nation. xx + 745pp. 5⅜ x 8. T468 Paperbound **$2.49**

FOUNDERS OF THE MIDDLE AGES, E. K. Rand. The best non-technical discussion of the transformation of Latin paganism into medieval civilization. Tertullian, Gregory, Jerome, Boethius, Augustine, the Neoplatonists, other crucial figures, philosophies examined. Excellent for the intelligent non-specialist. "Extraordinarily accurate," Richard McKeon, THE NATION. ix + 365pp. 5⅜ x 8. T369 Paperbound **$1.85**

THE POLITICAL THOUGHT OF PLATO AND ARISTOTLE, Ernest Barker. The standard, comprehensive exposition of Greek political thought. Covers every aspect of the "Republic" and the "Politics" as well as minor writings, other philosophers, theorists of the period, and the later history of Greek political thought. Unabridged edition. 584pp. 5⅜ x 8.
T521 Paperbound **$1.85**

PHILOSOPHY

THE GIFT OF LANGUAGE, M. Schlauch. (Formerly, "The Gift of Tongues.") A sound, middle-level treatment of linguistic families, word histories, grammatical processes, semantics, language taboos, word-coining of Joyce, Cummings, Stein, etc. 232 bibliographical notes. 350pp. 5⅜ x 8.
T243 Paperbound **$1.85**

THE PHILOSOPHY OF HEGEL, W. T. Stace. The first work in English to give a complete and connected view of Hegel's entire system. Especially valuable to those who do not have time to study the highly complicated original texts, yet want an accurate presentation by a most reputable scholar of one of the most influential 19th century thinkers. Includes a 14 x 20 fold-out chart of Hegelian system. 536pp. 5⅜ x 8. T254 Paperbound **$2.00**

ARISTOTLE, A. E. Taylor. A lucid, non-technical account of Aristotle written by a foremost Platonist. Covers life and works; thought on matter, form, causes, logic, God, physics, metaphysics, etc. Bibliography. New index compiled for this edition. 128pp. 5⅜ x 8.
T280 Paperbound **$1.00**

GUIDE TO PHILOSOPHY, C. E. M. Joad. This basic work describes the major philosophic problems and evaluates the answers propounded by great philosophers from the Greeks to Whitehead, Russell. "The finest introduction," BOSTON TRANSCRIPT. Bibliography, 592pp. 5⅜ x 8.
T297 Paperbound **$2.00**

LANGUAGE AND MYTH, E. Cassirer. Cassirer's brilliant demonstration that beneath both language and myth lies an unconscious "grammar" of experience whose categories and canons are not those of logical thought. Introduction and translation by Susanne Langer. Index. x + 103pp. 5⅜ x 8.
T51 Paperbound **$1.25**

SUBSTANCE AND FUNCTION, EINSTEIN'S THEORY OF RELATIVITY, E. Cassirer. This double volume contains the German philosopher's profound philosophical formulation of the differences between traditional logic and the new logic of science. Number, space, energy, relativity, many other topics are treated in detail. Authorized translation by W. C. and M. C. Swabey. xii + 465pp. 5⅜ x 8.
T50 Paperbound **$2.00**

THE PHILOSOPHICAL WORKS OF DESCARTES. The definitive English edition, in two volumes, of all major philosophical works and letters of René Descartes, father of modern philosophy of knowledge and science. Translated by E. S. Haldane and G. Ross. Introductory notes. Total of 842pp. 5⅜ x 8.
T71 Vol. 1, Paperbound **$2.00**
T72 Vol. 2, Paperbound **$2.00**

ESSAYS IN EXPERIMENTAL LOGIC, J. Dewey. Based upon Dewey's theory that knowledge implies a judgment which in turn implies an inquiry, these papers consider such topics as the thought of Bertrand Russell, pragmatism, the logic of values, antecedents of thought, data and meanings. 452pp. 5⅜ x 8.
T73 Paperbound **$1.95**

THE PHILOSOPHY OF HISTORY, G. W. F. Hegel. This classic of Western thought is Hegel's detailed formulation of the thesis that history is not chance but a rational process, the realization of the Spirit of Freedom. Translated and introduced by J. Sibree. Introduction by C. Hegel. Special introduction for this edition by Prof. Carl Friedrich, Harvard University. xxxix + 447pp. 5⅜ x 8.
T112 Paperbound **$1.85**

THE WILL TO BELIEVE and HUMAN IMMORTALITY, W. James. Two of James's most profound investigations of human belief in God and immortality, bound as one volume. Both are powerful expressions of James's views on chance vs. determinism, pluralism vs. monism, will and intellect, arguments for survival after death, etc. Two prefaces. 429pp. 5⅜ x 8.
T294 Clothbound **$3.75**
T291 Paperbound **$1.65**

INTRODUCTION TO SYMBOLIC LOGIC, S. Langer. A lucid, general introduction to modern logic, covering forms, classes, the use of symbols, the calculus of propositions, the Boole-Schroeder and the Russell-Whitehead systems, etc. "One of the clearest and simplest introductions," MATHEMATICS GAZETTE. Second, enlarged, revised edition. 368pp. 5⅜ x 8.
S164 Paperbound **$1.75**

MIND AND THE WORLD-ORDER, C. I. Lewis. Building upon the work of Peirce, James, and Dewey, Professor Lewis outlines a theory of knowledge in terms of "conceptual pragmatism," and demonstrates why the traditional understanding of the a priori must be abandoned. Appendices. xiv + 446pp. 5⅜ x 8.
T359 Paperbound **$1.95**

THE GUIDE FOR THE PERPLEXED, M. Maimonides One of the great philosophical works of all time, Maimonides' formulation of the meeting-ground between Old Testament and Aristotelian thought is essential to anyone interested in Jewish, Christian, and Moslem thought in the Middle Ages. 2nd revised edition of the Friedländer translation. Extensive introduction. lix + 414pp. 5⅜ x 8.
T351 Paperbound **$1.85**

DOVER BOOKS

THE PHILOSOPHICAL WRITINGS OF PEIRCE, J. Buchler, ed. (Formerly, "The Philosophy of Peirce.") This carefully integrated selection of Peirce's papers is considered the best coverage of the complete thought of one of the greatest philosophers of modern times. Covers Peirce's work on the theory of signs, pragmatism, epistemology, symbolic logic, the scientific method, chance, etc. xvi + 386pp. 5 3/8 x 8.
T216 Clothbound **$5.00**
T217 Paperbound **$1.95**

HISTORY OF ANCIENT PHILOSOPHY, W. Windelband. Considered the clearest survey of Greek and Roman philosophy. Examines Thales, Anaximander, Anaximenes, Heraclitus, the Eleatics, Empedocles, the Pythagoreans, the Sophists, Socrates, Democritus, Stoics, Epicureans, Sceptics, Neo-platonists, etc. 50 pages on Plato; 70 on Aristotle. 2nd German edition tr. by H. E. Cushman. xv + 393pp. 5 3/8 x 8.
T357 Paperbound **$1.75**

INTRODUCTION TO SYMBOLIC LOGIC AND ITS APPLICATIONS, R. Carnap. A comprehensive, rigorous introduction to modern logic by perhaps its greatest living master. Includes demonstrations of applications in mathematics, physics, biology. "Of the rank of a masterpiece," Z. für Mathematik und ihre Grenzgebiete. Over 300 exercises. xvi + 241pp. 5 3/8 x 8.
Clothbound **$4.00**
S453 Paperbound **$1.85**

SCEPTICISM AND ANIMAL FAITH, G. Santayana. Santayana's unusually lucid exposition of the difference between the independent existence of objects and the essence our mind attributes to them, and of the necessity of scepticism as a form of belief and animal faith as a necessary condition of knowledge. Discusses belief, memory, intuition, symbols, etc. xii + 314pp. 5 3/8 x 8.
T235 Clothbound **$3.50**
T236 Paperbound **$1.50**

THE ANALYSIS OF MATTER, B. Russell. With his usual brilliance, Russell analyzes physics, causality, scientific inference, Weyl's theory, tensors, invariants, periodicity, etc. in order to discover the basic concepts of scientific thought about matter. "Most thorough treatment of the subject," THE NATION. Introduction. 8 figures. viii + 408pp. 5 3/8 x 8.
T231 Paperbound **$1.95**

THE SENSE OF BEAUTY, G. Santayana. This important philosophical study of why, when, and how beauty appears, and what conditions must be fulfilled, is in itself a revelation of the beauty of language. "It is doubtful if a better treatment of the subject has since appeared," PEABODY JOURNAL. ix + 275pp. 5 3/8 x 8.
T238 Paperbound **$1.00**

THE CHIEF WORKS OF SPINOZA. In two volumes. Vol. I: The Theologico-Political Treatise and the Political Treatise. Vol. II: On the Improvement of Understanding, The Ethics, and Selected Letters. The permanent and enduring ideas in these works on God, the universe, religion, society, etc., have had tremendous impact on later philosophical works. Introduction. Total of 862pp. 5 3/8 x 8.
T249 Vol. I, Paperbound **$1.50**
T250 Vol. II, Paperbound **$1.50**

TRAGIC SENSE OF LIFE, M. de Unamuno. The acknowledged masterpiece of one of Spain's most influential thinkers. Between the despair at the inevitable death of man and all his works, and the desire for immortality, Unamuno finds a "saving incertitude." Called "a masterpiece," by the ENCYCLOPAEDIA BRITANNICA. xxx + 332pp. 5 3/8 x 8.
T257 Paperbound **$1.95**

EXPERIENCE AND NATURE, John Dewey. The enlarged, revised edition of the Paul Carus lectures (1925). One of Dewey's clearest presentations of the philosophy of empirical naturalism which reestablishes the continuity between "inner" experience and "outer" nature. These lectures are among the most significant ever delivered by an American philosopher. 457pp. 5 3/8 x 8.
T471 Paperbound **$1.85**

PHILOSOPHY AND CIVILIZATION IN THE MIDDLE AGES, M. de Wulf. A semi-popular survey of medieval intellectual life, religion, philosophy, science, the arts, etc. that covers feudalism vs. Catholicism, rise of the universities, mendicant orders, and similar topics. Bibliography. viii + 320pp. 5 3/8 x 8.
T284 Paperbound **$1.75**

AN INTRODUCTION TO SCHOLASTIC PHILOSOPHY, M. de Wulf. (Formerly, "Scholasticism Old and New.") Prof. de Wulf covers the central scholastic tradition from St. Anselm, Albertus Magnus, Thomas Aquinas, up to Suarez in the 17th century; and then treats the modern revival of scholasticism, the Louvain position, relations with Kantianism and positivism, etc. xvi + 271pp. 5 3/8 x 8.
T296 Clothbound **$3.50**
T283 Paperbound **$1.75**

A HISTORY OF MODERN PHILOSOPHY, H. Höffding. An exceptionally clear and detailed coverage of Western philosophy from the Renaissance to the end of the 19th century. Both major and minor figures are examined in terms of theory of knowledge, logic, cosmology, psychology. Covers Pomponazzi, Bodin, Boehme, Telesius, Bruno, Copernicus, Descartes, Spinoza, Hobbes, Locke, Hume, Kant, Fichte, Schopenhauer, Mill, Spencer, Langer, scores of others. A standard reference work. 2 volumes. Total of 1159pp. 5 3/8 x 8.
T117 Vol. 1, Paperbound **$2.00**
T118 Vol. 2, Paperbound **$2.00**

LANGUAGE, TRUTH AND LOGIC, A. J. Ayer. The first full-length development of Logical Positivism in English. Building on the work of Schlick, Russell, Carnap, and the Vienna school, Ayer presents the tenets of one of the most important systems of modern philosophical thought. 160pp. 5 3/8 x 8.
T10 Paperbound **$1.25**

ORIENTALIA AND RELIGION

THE MYSTERIES OF MITHRA, F. Cumont. The great Belgian scholar's definitive study of the Persian mystery religion that almost vanquished Christianity in the ideological struggle for the Roman Empire. A masterpiece of scholarly detection that reconstructs secret doctrines, organization, rites. Mithraic art is discussed and analyzed. 70 illus. 239pp. 5⅜ x 8.
T323 Paperbound $1.85

CHRISTIAN AND ORIENTAL PHILOSOPHY OF ART, A. K. Coomaraswamy. The late art historian and orientalist discusses artistic symbolism, the role of traditional culture in enriching art, medieval art, folklore, philosophy of art, other similar topics. Bibliography. 148pp. 5⅜ x 8.
T378 Paperbound $1.25

TRANSFORMATION OF NATURE IN ART, A. K. Coomaraswamy. A basic work on Asiatic religious art. Includes discussions of religious art in Asia and Medieval Europe (exemplified by Meister Eckhart), the origin and use of images in Indian art, Indian Medieval aesthetic manuals, and other fascinating, little known topics. Glossaries of Sanskrit and Chinese terms. Bibliography. 41pp. of notes. 245pp. 5⅜ x 8. T368 Paperbound $1.75

ORIENTAL RELIGIONS IN ROMAN PAGANISM, F. Cumont. This well-known study treats the ecstatic cults of Syria and Phrygia (Cybele, Attis, Adonis, their orgies and mutilatory rites); the mysteries of Egypt (Serapis, Isis, Osiris); Persian dualism; Mithraic cults; Hermes Trismegistus, Ishtar, Astarte, etc. and their influence on the religious thought of the Roman Empire. Introduction. 55pp. of notes; extensive bibliography. xxiv + 298pp. 5⅜ x 8.
T321 Paperbound $1.75

ANTHROPOLOGY, SOCIOLOGY, AND PSYCHOLOGY

PRIMITIVE MAN AS PHILOSOPHER, P. Radin. A standard anthropological work based on Radin's investigations of the Winnebago, Maori, Batak, Zuni, other primitive tribes. Describes primitive thought on the purpose of life, marital relations, death, personality, gods, etc. Extensive selections of original primitive documents. Bibliography. xviii + 420pp. 5⅜ x 8.
T392 Paperbound $2.00

PRIMITIVE RELIGION, P. Radin. Radin's thoroughgoing treatment of supernatural beliefs, shamanism, initiations, religious expression, etc. in primitive societies. Arunta, Ashanti, Aztec, Bushman, Crow, Fijian, many other tribes examined. "Excellent," NATURE. New preface by the author. Bibliographic notes. x + 322pp. 5⅜ x 8. T393 Paperbound $1.85

SEX IN PSYCHO-ANALYSIS, S. Ferenczi. (Formerly, "Contributions to Psycho-analysis.") 14 selected papers on impotence, transference, analysis and children, dreams, obscene words, homosexuality, paranoia, etc. by an associate of Freud. Also included: THE DEVELOPMENT OF PSYCHO-ANALYSIS, by Ferenczi and Otto Rank. Two books bound as one. Total of 406pp. 5⅜ x 8.
T324 Paperbound $1.85

THE PRINCIPLES OF PSYCHOLOGY, William James. The complete text of the famous "long course," one of the great books of Western thought. An almost incredible amount of information about psychological processes, the stream of consciousness, habit, time perception, memory, emotions, reason, consciousness of self, abnormal phenomena, and similar topics. Based on James's own discoveries integrated with the work of Descartes, Locke, Hume, Royce, Wundt, Berkeley, Lotse, Herbart, scores of others. "A classic of interpretation," PSYCHIATRIC QUARTERLY. 94 illus. 1408pp. 2 volumes. 5⅜ x 8.
T381 Vol. 1, Paperbound $2.50
T382 Vol. 2, Paperbound $2.50

THE POLISH PEASANT IN EUROPE AND AMERICA, W. I. Thomas, F. Znaniecki. Monumental sociological study of peasant primary groups (family and community) and the disruptions produced by a new industrial system and emigration to America, by two of the foremost sociologists of recent times. One of the most important works in sociological thought. Includes hundreds of pages of primary documentation; point by point analysis of causes of social decay, breakdown of morality, crime, drunkenness, prostitution, etc. 2nd revised edition. 2 volumes. Total of 2250pp. 6 x 9. T478 2 volume set, Clothbound $12.50

FOLKWAYS, W. G. Sumner. The great Yale sociologist's detailed exposition of thousands of social, sexual, and religious customs in hundreds of cultures from ancient Greece to Modern Western societies. Preface by A. G. Keller. Introduction by William Lyon Phelps. 705pp. 5⅜ x 8.
S508 Paperbound $2.49

BEYOND PSYCHOLOGY, Otto Rank. The author, an early associate of Freud, uses psychoanalytic techniques of myth-analysis to explore ultimates of human existence. Treats love, immortality, the soul, sexual identity, kingship, sources of state power, many other topics which illuminate the irrational basis of human existence. 291pp. 5⅜ x 8. T485 Paperbound $1.75

ILLUSIONS AND DELUSIONS OF THE SUPERNATURAL AND THE OCCULT, D. H. Rawcliffe. A rational, scientific examination of crystal gazing, automatic writing, table turning, stigmata, the Indian rope trick, dowsing, telepathy, clairvoyance, ghosts, ESP, PK, thousands of other supposedly occult phenomena. Originally titled "The Psychology of the Occult." 14 illustrations. 551pp. 5⅜ x 8.
T503 Paperbound $2.00

DOVER BOOKS

YOGA: A SCIENTIFIC EVALUATION, Kovoor T. Behanan. A scientific study of the physiological and psychological effects of Yoga discipline, written under the auspices of the Yale University Institute of Human Relations. Foreword by W. A. Miles, Yale Univ. 17 photographs. 290pp. 5⅜ x 8. T505 Paperbound **$1.65**

HOAXES, C. D. MacDougall. Delightful, entertaining, yet scholarly exposition of how hoaxes start, why they succeed, documented with stories of hundreds of the most famous hoaxes. "A stupendous collection . . . and shrewd analysis, "NEW YORKER. New, revised edition. 54 photographs. 320pp. 5⅜ x 8. T465 Paperbound **$1.75**

CREATIVE POWER: THE EDUCATION OF YOUTH IN THE CREATIVE ARTS, Hughes Mearns. Named by the National Education Association as one of the 20 foremost books on education in recent times. Tells how to help children express themselves in drama, poetry, music, art, develop latent creative power. Should be read by every parent, teacher. New, enlarged, revised edition. Introduction. 272pp. 5⅜ x 8. T490 Paperbound **$1.50**

LANGUAGES

NEW RUSSIAN-ENGLISH, ENGLISH-RUSSIAN DICTIONARY, M. A. O'Brien. Over 70,000 entries in new orthography! Idiomatic usages, colloquialisms. One of the few dictionaries that indicate accent changes in conjugation and declension. "One of the best," Prof. E. J. Simmons, Cornell. First names, geographical terms, bibliography, many other features. 738pp. 4½ x 6¼. T208 Paperbound **$2.00**

MONEY CONVERTER AND TIPPING GUIDE FOR EUROPEAN TRAVEL, C. Vomacka. Invaluable, handy source of currency regulations, conversion tables, tipping rules, postal rates, much other travel information for every European country plus Israel, Egypt and Turkey. 128pp. 3½ x 5¼. T260 Paperbound **60¢**

MONEY CONVERTER AND TIPPING GUIDE FOR TRAVEL IN THE AMERICAS (including the United States and Canada), **C. Vomacka.** The information you need for informed and confident travel in the Americas: money conversion tables, tipping guide, postal, telephone rates, etc. 128pp. 3½ x 5¼. T261 Paperbound **65¢**

DUTCH-ENGLISH, ENGLISH-DUTCH DICTIONARY, F. G. Renier. The most convenient, practical Dutch-English dictionary on the market. New orthography. More than 60,000 entries: idioms, compounds, technical terms, etc. Gender of nouns indicated. xviii + 571pp. 5½ x 6¼. T224 Clothbound **$2.50**

LEARN DUTCH!, F. G. Renier. The most satisfactory and easily-used grammar of modern Dutch. Used and recommended by the Fulbright Committee in the Netherlands. Over 1200 simple exercises lead to mastery of spoken and written Dutch. Dutch-English, English-Dutch vocabularies. 181pp. 4¼ x 7¼. T441 Clothbound **$1.75**

PHRASE AND SENTENCE DICTIONARY OF SPOKEN RUSSIAN, English-Russian, Russian-English. Based on phrases and complete sentences, rather than isolated words; recognized as one of the best methods of learning the idiomatic speech of a country. Over 11,500 entries, indexed by single words, with more than 32,000 English and Russian sentences and phrases, in immediately usable form. Probably the largest list ever published. Shows accent changes in conjugation and declension; irregular forms listed in both alphabetical place and under main form of word. 15,000 word introduction covering Russian sounds, writing, grammar, syntax. 15-page appendix of geographical names, money, important signs, given names, foods, special Soviet terms, etc. Travellers, businessmen, students, government employees have found this their best source for Russian expressions. Originally published as U.S. Government Technical Manual TM 30-944. iv + 573pp. 5⅝ x 8⅜. T496 Paperbound **$2.75**

PHRASE AND SENTENCE DICTIONARY OF SPOKEN SPANISH, Spanish-English, English-Spanish. Compiled from spoken Spanish, emphasizing idiom and colloquial usage in both Castilian and Latin-American. More than 16,000 entries containing over 25,000 idioms—the largest list of idiomatic constructions ever published. Complete sentences given, indexed under single words —language in immediately usable form, for travellers, businessmen, students, etc. 25-page introduction provides rapid survey of sounds, grammar, syntax, with full consideration of irregular verbs. Especially apt in modern treatment of phrases and structure. 17-page glossary gives translations of geographical names, money values, numbers, national holidays, important street signs, useful expressions of high frequency, plus unique 7-page glossary of Spanish and Spanish-American foods and dishes. Originally published as U.S. Government Technical Manual TM 30-900. iv + 513pp. 5⅝ x 8⅜. T495 Paperbound **$1.75**

SAY IT language phrase books

"SAY IT" in the foreign language of your choice! We have sold over ½ million copies of these popular, useful language books. They will not make you an expert linguist overnight, but they do cover most practical matters of everyday life abroad.

Over 1000 useful phrases, expressions, with additional variants, substitutions.

Modern! Useful! Hundreds of phrases not available in other texts: "Nylon," "air-conditioned," etc.

The ONLY inexpensive phrase book **completely indexed.** Everything is available at a flip of your finger, ready for use.

Prepared by native linguists, travel experts.

Based on years of travel experience abroad.

This handy phrase book may be used by itself, or it may supplement any other text or course; it provides a living element. Used by many colleges and institutions: Hunter College; Barnard College; Army Ordnance School, Aberdeen; and many others.

Available, 1 book per language:

Danish (T818) 75¢
Dutch T(817) 75¢
English (for German-speaking people) (T801) 60¢
English (for Italian-speaking people) (T816) 60¢
English (for Spanish-speaking people) (T802) 60¢
Esperanto (T820) 75¢
French (T803) 60¢
German (T804) 60¢
Modern Greek (T813) 75¢
Hebrew (T805) 60¢
Italian (T806) 60¢
Japanese (T807) 60¢
Norwegian (T814) 75¢
Russian (T810) 75¢
Spanish (T811) 60¢
Turkish (T821) 75¢
Yiddish (T815) 75¢
Swedish (T812) 75¢
Polish (T808) 75¢
Portuguese (T809) 75¢

LISTEN & LEARN language record sets

LISTEN & LEARN is the only language record course designed especially to meet your travel needs, or help you learn essential foreign language quickly by yourself, or in conjunction with any school course, by means of the automatic association method. Each set contains three 33⅓ rpm long-playing records — 1½ hours of recorded speech by eminent native speakers who are professors at Columbia, N.Y.U., Queens College and other leading universities. The sets are priced far below other sets of similar quality, yet they contain many special features not found in other record sets:

* Over 800 selected phrases and sentences, a basic vocabulary of over 3200 words.
* Both English and foreign language recorded; with a pause for your repetition.
* Designed for persons with limited time; no time wasted on material you cannot use immediately.
* Living, modern expressions that answer modern needs: drugstore items, "air-conditioned," etc.
* 128-196 page manuals contain everything on the records, plus simple pronunciation guides.
* Manual is fully indexed; find the phrase you want instantly.
* High fidelity recording—equal to any records costing up to $6 each.

The phrases on these records cover 41 different categories useful to the traveller or student interested in learning the living, spoken language: greetings, introductions, making yourself understood, passing customs, planes, trains, boats, buses, taxis, nightclubs, restaurants, menu items, sports, concerts, cameras, automobile travel, repairs, drugstores, doctors, dentists, medicines, barber shops, beauty parlors, laundries, many, many more.

"Excellent . . . among the very best on the market," Prof. Mario Pei, Dept. of Romance Languages, Columbia University. "Inexpensive and well-done . . . an ideal present," CHICAGO SUNDAY TRIBUNE. "More genuinely helpful than anything of its kind which I have previously encountered," Sidney Clark, well-known author of "ALL THE BEST" travel books. Each set contains 3 33⅓ rpm pure vinyl records, 128-196 page with full record text, and album. One language per set. LISTEN & LEARN record sets are now available in—

FRENCH	the set $4.95
ITALIAN	the set $4.95
RUSSIAN	the set $5.95
GERMAN	the set $4.95
SPANISH	the set $4.95
JAPANESE *	the set $5.95

* Available Sept. 1, 1959

UNCONDITIONAL GUARANTEE: Dover Publications stands behind every Listen and Learn record set. If you are dissatisfied with these sets for any reason whatever, return them within 10 days and your money will be refunded in full.

ART HISTORY

STICKS AND STONES, Lewis Mumford. An examination of forces influencing American architecture: the medieval tradition in early New England, the classical influence in Jefferson's time, the Brown Decades, the imperial facade, the machine age, etc. "A truly remarkable book," SAT. REV. OF LITERATURE. 2nd revised edition. 21 illus. xvii + 228pp. 5⅜ x 8.
T202 Paperbound **$1.60**

THE AUTOBIOGRAPHY OF AN IDEA, Louis Sullivan. The architect whom Frank Lloyd Wright called "the master," records the development of the theories that revolutionized America's skyline. 34 full-page plates of Sullivan's finest work. New introduction by R. M. Line. xiv + 335pp. 5⅜ x 8. T281 Paperbound **$1.85**

THE MATERIALS AND TECHNIQUES OF MEDIEVAL PAINTING, D. V. Thompson. An invaluable study of carriers and grounds, binding media, pigments, metals used in painting, al fresco and al secco techniques, burnishing, etc. used by the medieval masters. Preface by Bernard Berenson. 239pp. 5⅜ x 8. T327 Paperbound **$1.85**

PRINCIPLES OF ART HISTORY, H. Wölfflin. This remarkably instructive work demonstrates the tremendous change in artistic conception from the 14th to the 18th centuries, by analyzing 164 works by Botticelli, Dürer, Hobbema, Holbein, Hals, Titian, Rembrandt, Vermeer, etc., and pointing out exactly what is meant by "baroque," "classic," "primitive," "picturesque," and other basic terms of art history and criticism. "A remarkable lesson in the art of seeing," SAT. REV. OF LITERATURE. Translated from the 7th German edition. 150 illus. 254pp. 6⅛ x 9¼. T276 Paperbound **$2.00**

FOUNDATIONS OF MODERN ART, A. Ozenfant. Stimulating discussion of human creativity from paleolithic cave painting to modern painting, architecture, decorative arts. Fully illustrated with works of Gris, Lipchitz, Leger, Picasso, primitive, modern artifacts, architecture, industrial art, much more. 226 illustrations. 368pp. 6⅛ x 9¼. T215 Paperbound **$1.95**

HANDICRAFTS, APPLIED ART, ART SOURCES, ETC.

WILD FOWL DECOYS, J. Barber. The standard work on this fascinating branch of folk art, ranging from Indian mud and grass devices to realistic wooden decoys. Discusses styles, types, periods; gives full information on how to make decoys. 140 illustrations (including 14 new plates) show decoys and provide full sets of plans for handicrafters, artists, hunters, and students of folk art. 281pp. 7⅞ x 10¾. Deluxe edition. T11 Clothbound **$8.50**

METALWORK AND ENAMELLING, H. Maryon. Probably the best book ever written on the subject. Tells everything necessary for the home manufacture of jewelry, rings, ear pendants, bowls, etc. Covers materials, tools, soldering, filigree, setting stones, raising patterns, repoussé work, damascening, niello, cloisonné, polishing, assaying, casting, and dozens of other techniques. The best substitute for apprenticeship to a master metalworker. 363 photos and figures. 374pp. 5½ x 8½. T183 Clothbound **$7.50**

SHAKER FURNITURE, E. D. and F. Andrews. The most illuminating study of Shaker furniture ever written. Covers chronology, craftsmanship, houses, shops, etc. Includes over 200 photographs of chairs, tables, clocks, beds, benches, etc. "Mr. & Mrs. Andrews know all there is to know about Shaker furniture," Mark Van Doren, NATION. 48 full-page plates. 192pp. Deluxe cloth binding. 7⅞ x 10¾. T7 Clothbound **$6.00**

PRIMITIVE ART, Franz Boas. A great American anthropologist covers theory, technical virtuosity, styles, symbolism, patterns, etc. of primitive art. The more than 900 illustrations will interest artists, designers, craftworkers. Over 900 illustrations. 376pp. 5⅜ x 8.
T25 Paperbound **$1.95**

ON THE LAWS OF JAPANESE PAINTING, H. Bowie. The best possible substitute for lessons from an oriental master. Treats both spirit and technique; exercises for control of the brush; inks, brushes, colors; use of dots, lines to express whole moods, etc. 220 illus. 132pp. 6⅛ x 9¼. T30 Paperbound **$1.95**

HANDBOOK OF ORNAMENT, F. S. Meyer. One of the largest collections of copyright-free traditional art: over 3300 line cuts of Greek, Roman, Medieval, Renaissance, Baroque, 18th and 19th century art motifs (tracery, geometric elements, flower and animal motifs, etc.) and decorated objects (chairs, thrones, weapons, vases, jewelry, armor, etc.). Full text. 3300 illustrations. 562pp. 5⅜ x 8. T302 Paperbound **$2.00**

THREE CLASSICS OF ITALIAN CALLIGRAPHY. Oscar Ogg, ed. Exact reproductions of three famous Renaissance calligraphic works: Arrighi's OPERINA and IL MODO, Tagliente's LO PRESENTE LIBRO, and Palatino's LIBRO NUOVO. More than 200 complete alphabets, thousands of lettered specimens, in Papal Chancery and other beautiful, ornate handwriting. Introduction. 245 plates. 282pp. 6⅛ x 9¼. T212 Paperbound **$1.95**

THE HISTORY AND TECHNIQUES OF LETTERING, A. Nesbitt. A thorough history of lettering from the ancient Egyptians to the present, and a 65-page course in lettering for artists. Every major development in lettering history is illustrated by a complete alphabet. Fully analyzes such masters as Caslon, Koch, Garamont, Jenson, and many more. 89 alphabets, 165 other specimens. 317pp. 5⅜ x 8. T427 Paperbound **$2.00**

LETTERING AND ALPHABETS, J. A. Cavanagh. An unabridged reissue of "Lettering," containing the full discussion, analysis, illustration of 89 basic hand lettering tyles based on Caslon, Bodoni, Gothic, many other types. Hundreds of technical hints on construction, strokes, pens, brushes, etc. 89 alphabets, 72 lettered specimens, which may be reproduced permission-free. 121pp. 9¾ x 8. T53 Paperbound $1.25

THE HUMAN FIGURE IN MOTION, Eadweard Muybridge. The largest collection in print of Muybridge's famous high-speed action photos. 4789 photographs in more than 500 action-strip-sequences (at shutter speeds up to 1/6000th of a second) illustrate men, women, children—mostly undraped—performing such actions as walking, running, getting up, lying down, carrying objects, throwing, etc. "An unparalleled dictionary of action for all artists," AMERICAN ARTIST. 390 full-page plates, with 4789 photographs. Heavy glossy stock, reinforced binding with headbands. 7⅞ x 10¾. T204 Clothbound $10.00

ANIMALS IN MOTION, Eadweard Muybridge. The largest collection of animal action photos in print. 34 different animals (horses, mules, oxen, goats, camels, pigs, cats, lions, gnus, deer, monkeys, eagles—and 22 others) in 132 characteristic actions. All 3919 photographs are taken in series at speeds up to 1/1600th of a second, offering artists, biologists, cartoonists a remarkable opportunity to see exactly how an ostrich's head bobs when running, how a lion puts his foot down, how an elephant's knee bends, how a bird flaps his wings, thousands of other hard-to-catch details. "A really marvelous series of plates," NATURE. 380 full-pages of plates. Heavy glossy stock, reinforced binding with headbands. 7⅞ x 10¾. T203 Clothbound $10.00

THE BOOK OF SIGNS, R. Koch. 493 symbols—crosses, monograms, astrological, biological symbols, runes, etc.—from ancient manuscripts, cathedrals, coins, catacombs, pottery. May be reproduced permission-free. 493 illustrations by Fritz Kredel. 104pp. 6⅛ x 9¼. T162 Paperbound $1.00

A HANDBOOK OF EARLY ADVERTISING ART, C. P. Hornung. The largest collection of copyright-free early advertising art ever compiled. Vol. I: 2,000 illustrations of animals, old automobiles, buildings, allegorical figures, fire engines, Indians, ships, trains, more than 33 other categories! Vol II: Over 4,000 typographical specimens; 600 Roman, Gothic, Barnum, Old English faces; 630 ornamental type faces; hundreds of scrolls, initials, flourishes, etc "A remarkable collection," PRINTERS' INK.

Vol. I: Pictorial Volume. Over 2000 illustrations. 256pp. 9 x 12. T122 Clothbound $10.00

Vol. II: Typographical Volume. Over 4000 speciments. 319pp. 9 x 12. T123 Clothbound $10.00

Two volume set, Clothbound, only $18.50

DESIGN FOR ARTISTS AND CRAFTSMEN, L. Wolchonok. The most thorough course on the creation of art motifs and designs. Shows you step-by-step, with hundreds of examples and 113 detailed exercises, how to create original designs from geometric patterns, plants, birds, animals, humans, and man-made objects. "A great contribution to the field of design and crafts," N. Y. SOCIETY OF CRAFTSMEN. More than 1300 entirely new illustrations. xv + 207pp. 7⅞ x 10¾. T274 Clothbound $4.95

HANDBOOK OF DESIGNS AND DEVICES, C. P. Hornung. A remarkable working collection of 1836 basic designs and variations, all copyright-free. Variations of circle, line, cross, diamond, swastika, star, scroll, shield, many more. Notes on symbolism. "A necessity to every designer who would be original without having to labor heavily," ARTIST and ADVERTISER. 204 plates. 240pp. 5⅜ x 8. T125 Paperbound $1.90

THE UNIVERSAL PENMAN, George Bickham. Exact reproduction of beautiful 18th century book of handwriting. 22 complete alphabets in finest English roundhand, other scripts, over 2000 elaborate flourishes, 122 calligraphic illustrations, etc. Material is copyright-free. "An essential part of any art library, and a book of permanent value," AMERICAN ARTIST. 212 plates. 224pp. 9 x 13¾. T20 Clothbound $10.00

AN ATLAS OF ANATOMY FOR ARTISTS, F. Schider. This standard work contains 189 full-page plates, more than 647 illustrations of all aspects of the human skeleton, musculature, cutaway portions of the body, each part of the anatomy, hand forms, eyelids, breasts, location of muscles under the flesh, etc. 59 plates illustrate how Michelangelo, da Vinci, Goya, 15 others, drew human anatomy. New 3rd edition enlarged by 52 new illustrations by Cloquet, Barcsay. "The standard reference tool," AMERICAN LIBRARY ASSOCIATION. "Excellent," AMERICAN ARTIST. 189 plates, 647 illustrations. xxvi + 192pp. 7⅞ x 10⅝. T241 Clothbound $6.00

AN ATLAS OF ANIMAL ANATOMY FOR ARTISTS, W. Ellenberger, H. Baum, H. Dittrich. The largest, richest animal anatomy for artists in English. Form, musculature, tendons, bone structure, expression, detailed cross sections of head, other features, of the horse, lion, dog, cat, deer, seal, kangaroo, cow, bull, goat, monkey, hare, many other animals. "Highly recommended," DESIGN. Second, revised, enlarged edition with new plates from Cuvier, Stubbs, etc. 288 illustrations. 153pp. 11⅜ x 9. T82 Clothbound $6.00

ANIMAL DRAWING: ANATOMY AND ACTION FOR ARTISTS, C. R. Knight. 158 studies, with full accompanying text, of such animals as the gorilla, bear, bison, dromedary, camel, vulture, pelican, iguana, shark, etc., by one of the greatest modern masters of animal drawing. Innumerable tips on how to get life expression into your work. "An excellent reference work," SAN FRANCISCO CHRONICLE. 158 illustrations. 156pp. 10½ x 8½. T426 Paperbound $2.00

DOVER BOOKS

THE CRAFTSMAN'S HANDBOOK, Cennino Cennini. The finest English translation of IL LIBRO DELL' ARTE, the 15th century introduction to art technique that is both a mirror of Quattrocento life and a source of many useful but nearly forgotten facets of the painter's art. 4 illustrations. xxvii + 142pp. D. V. Thompson, translator. 6⅛ x 9¼. T54 Paperbound **$1.50**

THE BROWN DECADES, Lewis Mumford. A picture of the "buried renaissance" of the post-Civil War period, and the founding of modern architecture (Sullivan, Richardson, Root, Roebling), landscape development (Marsh, Olmstead, Eliot), and the graphic arts (Homer, Eakins, Ryder). 2nd revised, enlarged edition. Bibliography. 12 illustrations. xiv + 266 pp. 5⅜ x 8. T200 Paperbound **$1.65**

STIEGEL GLASS, F. W. Hunter. The story of the most highly esteemed early American glassware, fully illustrated. How a German adventurer, "Baron" Stiegel, founded a glass empire; detailed accounts of individual glasswork. "This pioneer work is reprinted in an edition even more beautiful than the original," ANTIQUES DEALER. New introduction by Helen McKearin. 171 illustrations, 12 in full color. xxii + 338pp. 7⅞ x 10¾. T128 Clothbound **$10.00**

THE HUMAN FIGURE, J. H. Vanderpoel. Not just a picture book, but a complete course by a famous figure artist. Extensive text, illustrated by 430 pencil and charcoal drawings of both male and female anatomy. 2nd enlarged edition. Foreword. 430 illus. 143pp. 6⅛ x 9¼. T432 Paperbound **$1.45**

PINE FURNITURE OF EARLY NEW ENGLAND, R. H. Kettell. Over 400 illustrations, over 50 working drawings of early New England chairs, benches, beds cupboards, mirrors, shelves, tables, other furniture esteemed for simple beauty and character. "Rich store of illustrations . . . emphasizes the individuality and varied design," ANTIQUES. 413 illustrations, 55 working drawings. 475pp. 8 x 10¾. T145 Clothbound **$10.00**

BASIC BOOKBINDING, A. W. Lewis. Enables both beginners and experts to rebind old books or bind paperbacks in hard covers. Treats materials, tools; gives step-by-step instruction in how to collate a book, sew it, back it, make boards, etc. 261 illus. Appendices. 155pp. 5⅜ x 8. T169 Paperbound **$1.35**

DESIGN MOTIFS OF ANCIENT MEXICO, J. Enciso. Nearly 90% of these 766 superb designs from Aztec, Olmec, Totonac, Maya, and Toltec origins are unobtainable elsewhere! Contains plumed serpents, wind gods, animals, demons, dancers, monsters, etc. Excellent applied design source. Originally $17.50. 766 illustrations, thousands of motifs. 192pp. 6⅛ x 9¼. T84 Paperbound **$1.85**

AFRICAN SCULPTURE, Ladislas Segy. 163 full-page plates illustrating masks, fertility figures, ceremonial objects, etc., of 50 West and Central African tribes—95% never before illustrated. 34-page introduction to African sculpture. "Mr. Segy is one of its top authorities," NEW YORKER. 164 full-page photographic plates. Introduction. Bibliography. 244pp. 6⅛ x 9¼. T396 Paperbound **$2.00**

THE PROCESSES OF GRAPHIC REPRODUCTION IN PRINTING, H. Curwen. A thorough and practical survey of wood, linoleum, and rubber engraving; copper engraving; drypoint, mezzotint, etching, aquatint, steel engraving, die sinking, stencilling, lithography (extensively); photographic reproduction utilizing line, continuous tone, photoengravure, collotype; every other process in general use. Note on color reproduction. Section on bookbinding. Over 200 illustrations, 25 in color. 143pp. 5½ x 8½. T512 Clothbound **$4.00**

CALLIGRAPHY, J. G. Schwandner. First reprinting in 200 years of this legendary book of beautiful handwriting. Over 300 ornamental initials, 12 complete calligraphic alphabets, over 150 ornate frames and panels, 75 calligraphic pictures of cherubs, stags, lions, etc., thousands of flourishes, scrolls, etc., by the greatest 18th century masters. All material can be copied or adapted without permission. Historical introduction. 158 full-page plates. 368pp. 9 x 13. T475 Clothbound **$10.00**

* * *

A DIDEROT PICTORIAL ENCYCLOPEDIA OF TRADES AND INDUSTRY, Manufacturing and the Technical Arts in Plates Selected from "L'Encyclopédie ou Dictionnaire Raisonné des Sciences, des Arts, et des Métiers," of Denis Diderot, edited with text by C. Gillispie. Over 2000 illustrations on 485 full-page plates. Magnificent 18th century engravings of men, women, and children working at such trades as milling flour, cheesemaking, charcoal burning, mining, silverplating, shoeing horses, making fine glass, printing, hundreds more, showing details of machinery, different steps in sequence, etc. A remarkable art work, but also the largest collection of working figures in print, copyright-free, for art directors, designers, etc. Two vols. 920pp. 9 x 12. Heavy library cloth. T421 Two volume set **$18.50**

* * *

SILK SCREEN TECHNIQUES, J. Biegeleisen, M. Cohn. A practical step-by-step home course in one of the most versatile, least expensive graphic arts processes. How to build an inexpensive silk screen, prepare stencils, print, achieve special textures, use color, etc. Every step explained, diagrammed. 149 illustrations, 8 in color. 201pp. 6⅛ x 9¼. T433 Paperbound **$1.45**

CATALOG OF

PUZZLES, GAMES, AND ENTERTAINMENTS

MATHEMATICS, MAGIC AND MYSTERY, Martin Gardner. Astonishing feats of mind reading, mystifying "magic" tricks, are often based on mathematical principles anyone can learn. This book shows you how to perform scores of tricks with cards, dice, coins, knots, numbers, etc., by using simple principles from set theory, theory of numbers, topology, other areas of mathematics, fascinating in themselves. No special knowledge required. 135 illus. 186pp. 5⅜ x 8.
T335 Paperbound $1.00

MATHEMATICAL PUZZLES FOR BEGINNERS AND ENTHUSIASTS, G. Mott-Smoth. Test your problem-solving techniques and powers of inference on 188 challenging, amusing puzzles based on algebra, dissection of plane figures, permutations, probabilities, etc. Appendix of primes, square roots, etc. 135 illus. 2nd revised edition. 248pp. 5⅜ x 8.
T198 Paperbound $1.00

LEARN CHESS FROM THE MASTERS, F. Reinfeld. Play 10 games against Marshall, Bronstein, Najdorf, other masters, and grade yourself on each move. Detailed annotations reveal principles of play, strategy, etc. as you proceed. An excellent way to get a real insight into the game. Formerly titled, "Chess by Yourself." 91 diagrams. vii + 144pp. 5⅜ x 8.
T362 Paperbound $1.00

REINFELD ON THE END GAME IN CHESS, F. Reinfeld. 62 end games of Alekhine, Tarrasch, Morphy, other masters, are carefully analyzed with emphasis on transition from middle game to end play. Tempo moves, queen endings, weak squares, other basic principles clearly illustrated. Excellent for understanding why some moves are weak or incorrect, how to avoid errors. Formerly titled, "Practical End-game Play." 62 diagrams. vi + 177pp. 5⅜ x 8.
T417 Paperbound $1.25

101 PUZZLES IN THOUGHT AND LOGIC, C. R. Wylie, Jr. Brand new puzzles you need no special knowledge to solve! Each one is a gem of ingenuity that will really challenge your problem-solving technique. Introduction with simplified explanation of scientic puzzle solving. 128pp. 5⅜ x 8.
T167 Paperbound $1.00

THE COMPLETE NONSENSE OF EDWARD LEAR. The only complete edition of this master of gentle madness at a popular price. The Dong with the Luminous Nose, The Jumblies, The Owl and the Pussycat, hundreds of other bits of wonderful nonsense. 214 limericks, 3 sets of Nonsense Botany, 5 Nonsense Alphabets, 546 fantastic drawings, much more. 320pp. 5⅜ x 8.
T167 Paperbound $1.00

28 SCIENCE FICTION STORIES OF H. G. WELLS. Two complete novels, "Men Like Gods" and "Star Begotten," plus 26 short stories by the master science-fiction writer of all time. Stories of space, time, future adventure that are among the all-time classics of science fiction. 928pp. 5⅜ x 8.
T265 Clothbound $3.95

SEVEN SCIENCE FICTION NOVELS, H. G. Wells. Unabridged texts of "The Time Machine," "The Island of Dr. Moreau," "First Men in the Moon," "The Invisible Man," "The War of the Worlds," "The Food of the Gods," "In the Days of the Comet." "One will have to go far to match this for entertainment, excitement, and sheer pleasure," N. Y. TIMES. 1015pp. 5⅜ x 8.
T264 Clothbound $3.95

MATHEMAGIC, MAGIC PUZZLES, AND GAMES WITH NUMBERS, R. V. Heath. More than 60 new puzzles and stunts based on number properties: multiplying large numbers mentally, finding the date of any day in the year, etc. Edited by J. S. Meyer. 76 illus. 129pp. 5⅜ x 8.
T110 Paperbound $1.00

FIVE ADVENTURE NOVELS OF H. RIDER HAGGARD. The master story-teller's five best tales of mystery and adventure set against authentic African backgrounds: "She," "King Solomon's Mines," "Allan Quatermain," "Allan's Wife," "Maiwa's Revenge." 821pp. 5⅜ x 8.
T108 Clothbound $3.95

WIN AT CHECKERS, M. Hopper. (Formerly "Checkers.") The former World's Unrestricted Checker Champion gives you valuable lessons in openings, traps, end games, ways to draw when you are behind, etc. More than 100 questions and answers anticipate your problems. Appendix. 75 problems diagrammed, solved. 79 figures. xi + 107pp. 5⅜ x 8.
T363 Paperbound $1.00

CRYPTOGRAPHY, L. D. Smith. Excellent introductory work on ciphers and their solution, history of secret writing, techniques, etc. Appendices on Japanese methods, the Baconian cipher, frequency tables. Bibliography. Over 150 problems, solutions. 160pp. 5⅜ x 8.
T247 Paperbound $1.00

CRYPTANALYSIS, H. F. Gaines. (Formerly, "Elementary Cryptanalysis.") The best book available on cryptograms and how to solve them. Contains all major techniques: substitution, transposition, mixed alphabets, multafid, Kasiski and Vignere methods, etc. Word frequency appendix. 167 problems, solutions. 173 figures. 236pp. 5⅜ x 8.
T97 Paperbound $1.95

FLATLAND, E. A. Abbot. The science-fiction classic of life in a 2-dimensional world that is considered a first-rate introduction to relativity and hyperspace, as well as a scathing satire on society, politics and religion. 7th edition. 16 illus. 128pp. 5⅜ x 8.
T1 Paperbound $1.00

DOVER BOOKS

HOW TO FORCE CHECKMATE, F. Reinfeld. (Formerly "Challenge to Chessplayers.") No board needed to sharpen your checkmate skill on 300 checkmate situations. Learn to plan up to 3 moves ahead and play a superior end game. 300 situations diagrammed; notes and full solutions. 111pp. 5⅜ x 8. T439 Paperbound **$1.25**

MORPHY'S GAMES OF CHESS, P. W. Sergeant, ed. Play forcefully by following the techniques used by one of the greatest chess champions. 300 of Morphy's games carefully annotated to reveal principles. Bibliography. New introduction by F. Reinfeld. 235 diagrams. x + 352pp. 5⅜ x 8. T386 Paperbound **$1.75**

MATHEMATICAL RECREATIONS, M. Kraitchik. Hundreds of unusual mathematical puzzlers and odd bypaths of math, elementary and advanced. Greek, Medieval, Arabic, Hindu problems; figurate numbers, Fermat numbers, primes; magic, Euler, Latin squares; fairy chess, latruncles, reversi, jinx, ruma, tetrachrome other positional and permutational games. Rigorous solutions. Revised second edition. 181 illus. 330pp. 5⅜ x 8. T163 Paperbound **$1.75**

MATHEMATICAL EXCURSIONS, H. A. Merrill. Revealing stimulating insights into elementary math, not usually taught in school. 90 problems demonstrate Russian peasant multiplication, memory systems for pi, magic squares, dyadic systems, division by inspection, many more. Solutions to difficult problems. 50 illus. 5⅜ x 8. T350 Paperbound **$1.00**

MAGIC TRICKS & CARD TRICKS, W. Jonson. Best introduction to tricks with coins, bills, eggs, ribbons, slates, cards, easily performed without elaborate equipment. Professional routines, tips on presentation, misdirection, etc. Two books bound as one: 52 tricks with cards, 37 tricks with common objects. 106 figures. 224pp. 5⅜ x 8. T909 Paperbound **$1.00**

MATHEMATICAL PUZZLES OF SAM LOYD, selected and edited by **M. Gardner.** 177 most ingenious mathematical puzzles of America's greatest puzzle originator, based on arithmetic, algebra, game theory, dissection, route tracing, operations research, probability, etc. 120 drawings, diagrams. Solutions. 187pp. 5⅜ x 8. T498 Paperbound **$1.00**

THE ART OF CHESS, J. Mason. The most famous general study of chess ever written. More than 90 openings, middle game, end game, how to attack, sacrifice, defend, exchange, form general strategy. Supplement on "How Do You Play Chess?" by F. Reinfeld. 448 diagrams. 356pp. 5⅜ x 8. T463 Paperbound **$1.85**

HYPERMODERN CHESS as Developed in the Games of its Greatest Exponent, ARON NIMZOVICH, F. Reinfeld, ed. Learn how the game's greatest innovator defeated Alekhine, Lasker, and many others; and use these methods in your own game. 180 diagrams. 228pp. 5⅜ x 8. T448 Paperbound **$1.35**

A TREASURY OF CHESS LORE, F. Reinfeld, ed. Hundreds of fascinating stories by and about the masters, accounts of tournaments and famous games, aphorisms, word portraits, little known incidents, photographs, etc., that will delight the chess enthusiast, captivate the beginner. 49 photographs (14 full-page plates), 12 diagrams. 315pp. 5⅜ x 8. T458 Paperbound **$1.75**

A NONSENSE ANTHOLOGY, collected by **Carolyn Wells.** 245 of the best nonsense verses ever written: nonsense puns, absurd arguments, mock epics, nonsense ballads, "sick" verses, dog-Latin verses, French nonsense verses, limericks. Lear, Carroll, Belloc, Burgess, nearly 100 other writers. Introduction by Carolyn Wells. 3 indices: Title, Author, First Lines. xxxiii + 279pp. 5⅜ x 8. T499 Paperbound **$1.25**

SYMBOLIC LOGIC and THE GAME OF LOGIC, Lewis Carroll. Two delightful puzzle books by the author of "Alice," bound as one. Both works concern the symbolic representation of traditional logic and together contain more than 500 ingenious, amusing and instructive syllogistic puzzlers. Total of 326pp. 5⅜ x 8. T492 Paperbound **$1.50**

PILLOW PROBLEMS and A TANGLED TALE, Lewis Carroll. Two of Carroll's rare puzzle works bound as one. "Pillow Problems" contain 72 original math puzzles. The puzzles in "A Tangled Tale" are given in delightful story form. Total of 291pp. 5⅜ x 8. T493 Paperbound **$1.50**

PECK'S BAD BOY AND HIS PA, G. W. Peck. Both volumes of one of the most widely read of all American humor books. A classic of American folk humor, also invaluable as a portrait of an age. 100 original illustrations. Introduction by E. Bleiler. 347pp. 5⅜ x 8. T497 Paperbound **$1.35**

Dover publishes books on art, music, philosophy, literature, languages, history, social sciences, psychology, handcrafts, orientalia, puzzles and entertainments, chess, pets and gardens, books explaining science, intermediate and higher mathematics mathematical physics, engineering, biological sciences, earth sciences, classics of science, etc. Write to:

Dept. catrr.
Dover Publications, Inc.
180 Varick Street, N. Y. 14, N. Y.